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ON THE CONVEX CASE IN THE POSITONE PROBLEM FOR ELLIPTIC SYSTEMS

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INTRODUCTION

IN THIS article we continue a study begun in [11] of the existence and dependence on the parameter λ of solutions of the system

$$L^\mu u^\mu = \lambda f^\mu(x, u) \quad \text{in } \Omega, \qquad u^\mu = 0 \quad \text{on } \partial\Omega, \qquad u^\mu > 0 \quad \text{in } \Omega, \tag{0.1}$$

$\mu = 1, \ldots, k$, where $u = (u^1, \ldots, u^k)$, $\Omega \subseteq \mathbb{R}^n$ is a smooth, bounded domain, and for each μ

$$L^\mu = -\sum_{i,j=1}^{n} a_{ij}^\mu(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i^\mu(x) \frac{\partial}{\partial x_i} + c^\mu(x) \tag{0.2}$$

is a uniformly elliptic differential operator with Hölder continuous coefficients, and $c^\mu \geqslant 0$. It is assumed that the map $f: \bar{\Omega} \times \mathbb{R}^k \to \mathbb{R}^k$, given by $f(x, w) = (f^1(x, w), \ldots, f^k(x, w))^T$ is continuous, that for each $x \in \bar{\Omega}$, $f(x, .)$ leaves the positive cone of $\mathbb{R}^k$ invariant, and that $f(x, 0) \not\equiv$ on $\bar{\Omega}$.

The system (0.1) and the assumptions following form a natural analogue to the problem

$$Lu = \lambda f(x, u) \quad \text{in } \Omega, \qquad u \equiv 0 \quad \text{on } \partial\Omega, \qquad u > 0 \quad \text{in } \Omega, \tag{0.3}$$

where L is as in (0.2) and $f: \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous, cone preserving, and $f(x, 0) > 0$ for $x \in \Omega$.

The problem (0.3) has been the object of considerable study, beginning with the paper of Cohen and Keller [12] in 1967. (The results of [12] are also presented in [22], with some additions and corrections.) Subsequently, the results of [12] have been extended in various directions.

Initially, Cohen and Keller [12] developed a monotone scheme for finding the minimal solution of (0.3), showed that the set of parameter values λ for which such solutions exist is an interval $(0, \lambda^*)$ or $(0, \lambda^*]$ (λ^* can possibly be $+\infty$), gave estimates for λ^* in terms of the first eigenvalue of related linear problems, and showed that solutions to (0.3) are unique in case f is concave. Monotone methods such as those employed in [12] have been widely used in the study of elliptic and parabolic problems; a unified treatment of such methods in a fairly general setting is given by Amann in [3]. In [11], the authors of this article extended most of the results of [12] to weakly coupled systems of the form (0.1). Similar results for an abstract equation are given in [24].

In the study of (0.1) and (0.3), the theory of linear eigenvalue problems plays a central role. For a single equation, the necessary theory has been available since the late 1950's; see [12]. For the system case, some results can be obtained from the general theory of positive operators on ordered Banach spaces, as is done in [3]. However, for a problem such as

$$\begin{aligned} L^\alpha u^\alpha &= \lambda \sum_{\beta=1}^{k} m^{\alpha\beta}(x) u^\beta \quad \text{in } \Omega \\ u &= 0 \quad\quad\quad\quad\quad\quad\;\; \text{on } \partial\Omega, \end{aligned} \tag{0.4}$$

$\alpha = 1, \ldots, k$, the theory of [3] requires $m^{\alpha\beta}(x) > 0$ in $\bar{\Omega}$ for all $\alpha, \beta = 1, \ldots, k$. This condition is too restrictive for our purposes; to avoid it in this paper and in [11], we have used the more specialized but more delicate analysis of (0.4) begun in [20] and extended and developed in [10].

For the case of concave nonlinearities, the results of [12] and [11] give a reasonably complete theory for (0.3) and (0.1), respectively. The case of convex and/or asymptotically linear nonlinearities is more complicated, since multiple solutions may occur. A survey of much of the theory for (0.3) is given in [25], along with a partial review of the literature. In the case of convex nonlinearities, it is shown in [11, 12] that if λ^* is the upper endpoint of the interval in λ for which solutions exist, then λ^* is finite.

A number of the results which have been established for (0.3) in the convex case deal with the question of multiplicity of solutions. Analogues to these results for (0.1) are the primary concern of this paper. The first type of result giving multiple solutions asserts that if the minimal solutions to (0.3) are bounded on $(0, \lambda^*)$, then a solution exists for $\lambda = \lambda^*$ and at least two solutions exist on $(\lambda^* - \varepsilon, \lambda^*)$ for some $\varepsilon > 0$. Such results for (0.3) were obtained by Keener and Keller [21], Bandle [7], and Crandall and Rabinowitz [16]; see also the article of Mignot and Puel [27]. In [21], the existence of multiple solutions for $\lambda \in (0, \lambda^*)$ is obtained by perturbed bifurcation theory; in [7, 16, 27] via arguments based on the implicit function theorem. (A similar result using the implicit function theorem is given by Amann in [1] and applied to the asymptotically linear case.) In such results it is necessary to obtain *a priori* bounds on the minimal solution. In the present article we obtain results for (0.1) similar to those found for (0.3). We obtain *a priori* bounds on the minimal solution via an adaptation of the methods of [16], then use an abstract result given in [1] to obtain the existence of multiple solutions. The methods of [16] must be modified somewhat as they involve the variational characterization of eigenvalues, and for systems the requirement of a variational structure for the nonlinearity imposes unreasonable restrictions; see the discussion in [13]. Several of the results of [7, 16], and [27] depend on variational methods, and so do not extend readily to systems where the nonlinearity is not a gradient. In [7] isoperimetric inequalities are used to obtain *a priori* bounds; we have not attempted to use this approach for systems.

The second type of result on multiple solutions asserts that if *all* positive solutions to (0.3) are uniformly bounded for $\lambda \in [\lambda_1, \lambda^*]$ $(\lambda_1 > 0)$, then there exist at least two solutions for $\lambda \in [\lambda_1, \lambda^*)$. These results differ from those discussed in the preceding paragraph by providing specific information on the values of λ for which multiple solutions exist. The existence of at least two solutions for λ in the interval $[\lambda_1, \lambda^*)$ follows from topological arguments based on degree theory or the fixed point index of cone maps. Such results are given in [1, 3, 17, 18, 26]. The same abstract results apply to (0.1), provided *a priori* estimates are available. There are two cases which occur in the above-mentioned papers: superlinear and asymptotically linear nonlinearities. In the superlinear case, *a priori* estimates for solutions to (0.3) are derived in [9, 18, 25] and for solutions to (0.1) in [13, 18]. These estimates hold for $\lambda \in [\lambda_1, \lambda^*]$ for any $\lambda_1 > 0$; so we have at least two solutions for $\lambda \in (0, \lambda^*)$. The estimate for (0.3) imposing the weakest growth restriction on the nonlinearity is given in [18], but that result does not extend easily to systems as it involves an antiderivative of the nonlinearity; such an antiderivative is only available for systems where the nonlinearity is a gradient. We use the estimates of [13], which extend to systems those of [9] and [25]. For the asymptotically linear case, a rather general abstract theory is developed by Amann in [1] (see also [3]). For (0.1) to be asymptotically linear, we must have

$$\frac{\partial f(x,u)}{\partial u} \to M(x) \quad \text{as } u \to \infty,$$

for some matrix $M(x)$; to apply the results of [1] directly requires that

$$L^{-1}M(x)\left(L = \begin{pmatrix} L_1 & & \\ & \ddots & \\ & & L_k \end{pmatrix}, \text{ see (0.1)–(0.2)}\right)$$

be strongly positive, which requires in turn that each entry of $M(x)$ be positive. To avoid this restriction, we rework some of the analysis of [1] using results from [10] to replace strong positively. In the asymptotically linear case, *a priori* bounds follow from the general theory; they hold on $[\lambda_1, \lambda^*]$ for any $\lambda_1 > \lambda_\infty$, where λ_∞ is the first eigenvalue of the problem

$$\begin{aligned} Lu &= \lambda M(x)u \quad && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{0.5}$$

where $f_u(x, u) \to M(x)$ as $u \to \infty$. Hence (0.3) or (0.1) has at least two solutions for $\lambda \in (\lambda_\infty, \lambda^*)$. Such results for (0.3) are obtained in [1, 4]. Results implying the existence of exactly two solutions on a subinterval of $(\lambda_\infty, \lambda^*)$ are given in [2, 5, 30]. They depend on the self-adjointness of the problem (0.3), which does not hold in general for systems. In [17], some of the abstract results of [1] are improved, and in [6] the case of (0.3) when f need not be positive is treated.

In this article, a treatment of both types of multiplicity results is given. We give the topological arguments in Section 2, while in Section 3, we present the necessary *a priori* estimates.

Various investigators have obtained results for (0.2) other than lower bounds on the number of solutions. In [7, 8] estimates for λ^* and for solutions to (0.3) are derived; in [7, 16, 27] the stability of minimal and other solutions to (0.3) (viewed as steady states of a parabolic equation) is analyzed. We do not consider such questions for (0.1) in this article. Other estimates for (0.3) or (0.1) are given in [14]. In [21, 23], it is shown that under appropriate hypotheses, (0.3) cannot have three distinct solutions u_1, u_2, u_3 with $0 < u_1 \leq u_2 \leq u_3$. In Section 4, we extend this result to (0.1), along with a uniqueness result for $\lambda \in (0, \lambda_\infty)$ in the asymptotically linear case to (0.1). The corresponding result for (0.3) may be found in [4].

Study of (0.1) leads naturally to study of the multiparameter system

$$\begin{aligned} L^\mu u^\mu &= \lambda^\mu f^\mu(x, u) \quad && \text{in } \Omega \\ u^\mu &= 0 && \text{on } \partial\Omega \\ u^\mu &> 0 && \text{in } \Omega, \end{aligned} \tag{0.6}$$

where L and f are as before. Corresponding multiplicity results for (0.6) are obtained in Sections 2 and 3. In fact, much more is shown. As in [18], our multiplicity results for (0.1) are given in terms of continua (closed connected sets) of solutions. In the case (0.6), these solution continua will be of topological dimension at least k. We obtain this result via the fixed point index and the global multidimensional implicit function theorem of Fitzpatrick, Massabo, and Pejsachowitz [19]. (For a precise definition of topological dimension, see [19].) We also begin an examination of another purely system phenomena that arises with the study of (0.6). Namely, we give some results in Section 4 on the shape of the region $\bar{\Lambda}$ in the positive cone in $\mathbb{R}^k$ such that (0.6) has solutions for $(\lambda^1, \ldots, \lambda^k) \in \bar{\Lambda}$. Our results are a natural start towards deeper examination of the effects of convexity and coupling in (0.1) as well as (0.6). Finally, we note that for the convenience of the reader, the various hypotheses to be placed on the nonlinearity f as well as principal results from [11] are collected in Section 1.

1. PRELIMINARY RESULTS

Throughout this paper, we will not explicitly distinguish vectors from scalars by notation. Which type of quantity a variable represents will either be stated or will follow readily from context. In our analysis, we will make extensive use of the notion of ordering. Our notation can best be described in terms of ordered Banach spaces. Let E be an ordered Banach space with positive cone P; that is, let E be a real Banach space and let $P \subseteq E$ be such that $\mathbb{R}^+ P \subseteq P$, $P + P \subseteq P$, and $P \cap (-P) = \{0\}$. If $u, v \in E$, we write $u \geqslant v$ if $u - v \in P$, and $u > v$ if $u - v \in P \backslash \{0\}$, For Euclidean spaces $\mathbb{R}^m$ we take $P = (\bar{\mathbb{R}}^+)^m$; for function spaces (typically Sobolev or Hölder spaces) we take $P = \{u \in E : u(x) \geqslant 0 \text{ on } \bar{\Omega}\}$. We will also use a more specialized notation: if $u, v \in C^1(\bar{\Omega})$, we will write $u \gg v$ if $u(x) \geqslant v(x)$ on $\bar{\Omega}$, $u(x) > v(x)$ on Ω, and $(\partial u/\partial \eta) < (\partial v/\partial \eta)$ on $\partial\Omega$. If $u, v \in [C^1(\bar{\Omega})]^k$ we write $u \gg v$ if $u^\mu \gg v^\mu$ for all components of u and v. (Note that $u \gg 0$ does not imply $u \in \operatorname{int} P$, since we may have $u = 0$ on $\partial\Omega$. However, if $u \gg 0$ and $v \in [C^1(\bar{\Omega})]^k$ with $v = 0$ on $\partial\Omega$, then $u \pm v \gg 0$ if $\|v\|$ is sufficiently small. Since we will be dealing with Dirichlet boundary conditions, the notation "$\gg$" will be quite useful.)

We are now in a position to collect the various hypotheses to be placed on the nonlinearity f in this article. We will assume f is C^2, but that is not needed for all the results. The reader will find it convenient to refer to the following list while reading this paper:

(H1) $f(x, .) : (\bar{\mathbb{R}}^+)^k \to (\bar{\mathbb{R}}^+)^k$ for all $x \in \bar{\Omega}$;

(H2) For each $\mu \in \{1, \ldots, k\}$ there is a sequence $\nu_0, \nu_1, \ldots, \nu_N$ in $\{1, \ldots, k\}$ with $\nu_N = \mu$ such that $f^{\nu_0}(x, 0) > 0$ for some $x \in \Omega$, and if $u : \bar{\Omega} \to (\bar{\mathbb{R}}^+)^k$ with $u^{\nu_j}(x) > 0$ on Ω for $j = 0, \ldots, J$, $J \leqslant N - 1$, then $f^{\nu_{J+1}}(x, u) > 0$ for some $x \in \Omega$;

(H3) $f^\mu(x, u)$ is nondecreasing in u^ν for $\nu \neq \mu$;

(H4) There is a constant $K \geqslant 0$ such that for all μ, if $u^\mu = \bar{u}^\nu \geqslant 0$ for $\nu \neq \mu$ and $u^\mu \geqslant \bar{u}^\mu$, then $f^\mu(x, u) - f^\mu(x, \bar{u}) \geqslant -K(u^\mu - \bar{u}^\mu)$;

(H5) If $u \geqslant 0$, $\partial f^\alpha(x, u)/\partial u^\beta \geqslant 0$ for $\alpha, \beta = 1, \ldots, k$, for all $x \in \bar{\Omega}$;

(H6) If $u \gg 0$, $f_u(x, u)$ is a nonnegative irreducible matrix with

$$\frac{\partial f^\alpha(x, u)}{\partial u^\alpha} > 0$$

for $\alpha = 1, \ldots, k$;

(H7) $\dfrac{\partial f^\alpha}{\partial u^\beta}(x, w) \geqslant \dfrac{\partial f^\alpha}{\partial u^\beta}(x, w')$ for $w \gg w' \geqslant 0$ if $\alpha, \beta = 1, \ldots k$, with strict inequality if $\alpha = \beta$;

(H8) $\left(\dfrac{\partial^2 f^\alpha}{\partial u^\beta\, \partial u^\gamma}(x, w)\right)_{\beta,\gamma=1}^{k}$

is positive semi-definite for $x \in \bar{\Omega}$ and $w \geqslant 0$, for each $\alpha = 1, 2, \ldots, k$.

Note. If f satisfies (H1)–(H2) and (H5)–(H8), f is said to be *convex*.

This article is a continuation of the work begun in [11], and, as such, the results of [11] are freely used throughout this paper. However, in order that this article be somewhat self-contained, we list some of the significant results of [11]. (The reader should note that numbering in this list is that of [11] and is not to be confused with that of the present article.) Proofs are found in [11]. In what follows, $\Lambda = \{\lambda > 0\colon (0.1)$ has a solution $u \gg 0\}$.

LEMMA 2.1 [11]. Let $M(x)$ be a $k \times k$ matrix with $m^{\mu\nu}$ of class $C^\alpha(\bar{\Omega})$ and $m^{\mu\nu}(x) \geqslant 0$ for all $x \in \bar{\Omega}$. Let $h\colon \bar{\Omega} \to (\bar{\mathbb{R}}^+)^k$ be of class $C^\alpha(\bar{\Omega})$. Consider the system

$$\begin{aligned} Lu &= \lambda M u + h, \qquad & x &\in \Omega \\ u &= 0, & x &\in \partial\Omega, \end{aligned} \tag{*}$$

where

$$L = \begin{pmatrix} L^1 & & 0 \\ & \ddots & \\ 0 & & L^k \end{pmatrix}, \quad u = \begin{pmatrix} u^1 \\ \vdots \\ u^k \end{pmatrix},$$

and for each μ, $u^\mu \in C^{2+\alpha}(\bar{\Omega})$.

(i) Suppose there is $x_0 \in \Omega$ and $\mu \in \{1, \ldots, k\}$ such that $m^{\mu\mu}(x_0) > 0$. Then the system

$$\begin{aligned} Lv &= \lambda M v, \qquad & x &\in \Omega \\ v &\equiv 0, & x &\in \partial\Omega \end{aligned} \tag{**}$$

has a smallest positive eigenvalue λ_0 admitting a nonnegative solution v. Furthermore, if $\lambda < \lambda_0$, (*) has a unique nonnegative solution for any $h \geqslant 0$. If $\lambda \geqslant \lambda_0$, (*) has no nonnegative solution provided $h^\mu(x^\mu) > 0$ for some $x_\mu \in \Omega$, $\mu = 1, \ldots, k$.

(ii) If, in addition, there is $x_0 \in \Omega$ such that $M(x_0)$ is irreducible and $m^{\mu\mu}(x_0) > 0$ for $\mu = 1, \ldots, k$, (*) has a nonnegative solution for $h > 0$ only in case $\lambda < \lambda_0$. In this case, the solution u is such that $u \gg 0$.

COROLLARY 2.3 [11]. Suppose that M has the form

$$M = \begin{pmatrix} M^1 & 0 & \cdots & 0 \\ 0 & M^2 & \cdots & \cdot \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & M^N \end{pmatrix}$$

where M^γ is an $r_\gamma \times r_\gamma$ matrix satisfying the hypotheses of lemma 2.1 (ii) [11], with $\sum_{r=1}^{N} r_\gamma = k$. Let $\lambda_0(M^\gamma)$ denote the smallest eigenvalue of the system

$$L^\gamma w = \lambda M^\gamma w \tag{***}$$

where

$$L^\gamma = \begin{pmatrix} L^{\delta_\gamma+1} & 0 & \\ 0 & \ddots & \\ \vdots & & L^{\delta_\gamma+r_\gamma} \end{pmatrix}$$

and

$$\delta_\gamma = \begin{cases} 0, & \text{if } \gamma = 1 \\ \sum_{j=1}^{\gamma-1} r_j, & \text{if } 2 \leq \gamma \leq N. \end{cases}$$

Then if $h^{\delta_\gamma+m} \neq 0$ for some $m \in \{1, \ldots, k_\gamma\}$, (*) has a nontrivial solution u with

$$\begin{pmatrix} u^{\delta_\gamma+1} \\ \cdot \\ \cdot \\ \cdot \\ u^{\delta_\gamma+r_\gamma} \end{pmatrix} \neq 0 \text{ only if } \lambda < \lambda_0(M^\gamma).$$

Furthermore, in this case, $u^{\delta_\gamma+m}(x) > 0$ for $x \in \Omega$, and $m = 1, \ldots, r_\gamma$.

THEOREM 3.1 [11]. Suppose that $f(x, u)$ satisfies (H1)–(H4). The iteration scheme

$$\begin{aligned} &u_0 = 0 \\ &Lu_{n+1} + \lambda K u_{n+1} = \lambda(f(x, u_n) + K u_n) \quad \text{in } \Omega \\ &u_{n+1} = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{****}$$

produces a sequence which is increasing in each component. A number $\lambda > 0$ belongs to Λ if and only if the sequence $\{u_n\}$ is uniformly bounded; in that case, the sequence converges uniformly to a solution $\underline{u}(\lambda, x)$ of (0.1). The solution $\underline{u}(\lambda, x)$ is minimal in the sense that $\underline{u}(\lambda, x) \leq u(x)$ for any other positive solution of (0.1).

THEOREM 3.3[11]. Suppose that $f(x, u)$ satisfies (H1)–(H4), and $\lambda' \in \Lambda$. Then $(0, \lambda') \subseteq \Lambda$. Further, $\underline{u}(\lambda, x)$ is strictly increasing in λ. (It follows that Λ is an interval.)

COROLLARY 3.4[11]. Let $\lambda^* = \sup \Lambda$. Suppose that f satisfies (H1)–(H4) and there exist a function $h: \bar{\Omega} \to (\mathbb{R}^+)^k$ and matrix $M = (m^{\mu\nu}(x))$ satisfying the hypotheses of (i) of lemma 2.1[11] such that for each u, v with $v \geqslant u \geqslant 0$, we have for $x \in \Omega$, $\mu = 1, \ldots, k$, that

$$f^\mu(x, u) + Ku^\mu \leqslant h^\mu(x) + \sum_{\nu=1}^{k} m^{\mu\nu}(x)v^\nu + Kv^\mu.$$

Then $\lambda^* \geqslant \lambda_0(M)$ where $\lambda_0(M)$ is the first eigenvalue for (**).

THEOREM 4.1[11]. Let $f(x, u)$ satisfy (H1), (H2), (H5) and (H6), and let $\lambda^* = \sup \Lambda$. Then for each $\lambda \in (0, \lambda^*)$, $\lambda \leqslant \mu_1(\lambda)$, where $\mu_1(\lambda) = \mu_1(f_u(x, \underline{u}(\lambda, x)))$ is the principal eigenvalue of

$$\begin{aligned} L\psi &= \mu f_u(x, \underline{u}(\lambda, x))\psi, & x &\in \Omega \\ \psi &\equiv 0 & x &\in \partial\Omega. \end{aligned}$$

COROLLARY 4.2[11]. Suppose that f is convex. Then $\mu_1(\lambda)$ is a decreasing function on λ on $(0, \lambda^*)$, and, furthermore, $\mu_1(\lambda) > \lambda^* > \lambda$.

LEMMA 4.3[11]. Suppose that $f(x, u)$ satisfies (H1), (H2), (H5) and (H6), that $\lambda^* < \infty$, and there exists a constant M such that $\sup_{x \in \bar{\Omega}} |\underline{u}(\lambda, x)| < M$ for $\lambda \in \Lambda$. Then $\lambda^* \in \Lambda$, and $\underline{u}(\lambda, x) \to \underline{u}(\lambda^*, x)$ as $\lambda \uparrow \lambda^*$.

2. MULTIPLICITY RESULTS

In this section, we give some results on the multiplicity of solutions for (0.1) and (0.6) in the convex case. We begin with the following simple general observation, which follows as in [1, theorem 4.1].

PROPOSITION 2.1. Let $\Sigma = \{(\lambda, u) \in [0, \infty) \times [C_0^\alpha(\bar{\Omega})]^k : (\lambda, u)$ solves (0.1) and $u \geqslant 0\}$. Then Σ has an unbounded component $\mathscr{C}$ emanating from $\{(0, 0)\}$.

We now consider $\mathscr{C}$ in more detail in case f is convex.

LEMMA 2.2. If f is convex and $u(\lambda)$ denotes the minimal positive solution of (0.1) for $\lambda \in [0, \lambda^*)$ (where $\underline{u}(0) \equiv 0$), then $\lambda \to \underline{u}(\lambda)$ is continuous as a map from $[0, \lambda^*)$ to $[C_0^\alpha(\bar{\Omega})]^k$, where $0 < \alpha < 1$.

Proof. From the proof of theorem 4.1 of [11], it suffices to establish continuity from the right. Suppose then that $0 \leqslant \lambda' < \lambda^*$ and let $u' = u(\lambda')$. Pick $\bar{\lambda} \in (\lambda', \lambda^*)$ and let $u = \underline{u}(\bar{\lambda})$. If $\lambda \in (\lambda, \bar{\lambda})$ and $w = \underline{u}(\lambda)$,

$$\begin{aligned} L(w - u') &= \lambda f(x, w) - \lambda' f(x, u') \\ &= (\lambda - \lambda')f(x, u') + \lambda[f(x, w) - f(x, u')] \\ &= (\lambda - \lambda')f(x, u') + \lambda\left[\int_0^1 f_u(x, \theta w + (1 - \theta)u')\, d\theta\right] \cdot (w - u'). \end{aligned}$$

By theorem 3.3 of [11], $u' \leqslant w \leqslant \bar{u}$. Since f is convex,

$$\left[\int_0^1 f_u(x, \theta w + (1-\theta)u')\, d\theta\right] \cdot (w - u') \leqslant f_u(x, \bar{u}) \cdot (w - u')$$

and $\bar{\lambda} < \lambda^* < \mu_1(\bar{\lambda})$. Furthermore, $[L - \lambda f_u(x, \bar{u})]^{-1}$ exists as a compact positive operator on $[C_0^\alpha(\bar{\Omega})]^k$, which is continuously dependent on $\lambda \in [\lambda', \bar{\lambda}]$ in the strong operator topology. Hence

$$0 \leqslant w - u' \leqslant (\lambda - \lambda')[L - \lambda f_u(x, \bar{u})]^{-1} \cdot f(x, u').$$

Thus $w \to u'$ in $[C_0(\bar{\Omega})]^k$ and hence via standard elliptic theory, in $[C_0^\alpha(\bar{\Omega})]^k$, (as in [11]).
Let $M = \sup_{\lambda \in [0, \lambda^*)} \|\underline{u}(\lambda)\|_{[C(\bar{\Omega})]^k}$.

THEOREM 2.3. Suppose $M < \infty$ and that f is convex. Then

(i) There exists an unique positive solution to (0.1) for $\lambda = \lambda^*$.
(ii) There exists $\phi \gg 0$ such that

$$\begin{aligned} L\phi &= \lambda^* f_u(x, \underline{u}(\lambda^*))\phi && \text{in } \Omega \\ \phi &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Furthermore, λ^* is a simple eigenvalue of (2.1).

(iii) There exists a neighborhood V of $(\lambda^*, \underline{u}(\lambda^*))$ in $[0, \infty) \times [C_0^{2+\alpha}(\bar{\Omega})]^k$, $\varepsilon > 0$; and a C^2 map $\sigma \to (\lambda(\sigma), \underline{u}(\sigma))$ from $(-\varepsilon, \varepsilon) \to V$ such that $(\lambda(0), \underline{u}(0)) = (\lambda^*, \underline{u}(\lambda^*))$ and such that $(\lambda, u) \in V \cap \Sigma$ implies $(\lambda, u) = (\lambda(\sigma), \underline{u}(\sigma))$ for some $\sigma \in (-\varepsilon, \varepsilon)$.

Proof. The existence of a solution of (0.1) for $\lambda = \lambda^*$ follows as in lemma 4.3 of [11]. Observe that $\underline{u}(\lambda) \to \underline{u}(\lambda^*)$ in $[C_0^{2+\alpha}(\bar{\Omega})]^k$ by lemma 2.2 and standard elliptic theory.

Define a map $H: \mathbb{R} \times [C_0^{2+\alpha}(\bar{\Omega})]^k \to [C_0^{2+\alpha}(\bar{\Omega})]^k$ by $H(\lambda, u) = u - \lambda L^{-1} f(x, u)$. Note that $H_u(\lambda, u)w = w - \lambda L^{-1} f_u(x, u)w$. Then $H_{\underline{u}}(\lambda^*, \underline{u}(\lambda^*))$ is singular since otherwise the implicit function theorem implies there exists $\bar{\lambda} > \lambda^*$ such that (0.1) has a positive solution for $\lambda = \bar{\lambda}$.

If $\lambda < \lambda^*$, $\mu_1(\lambda) > \lambda^*$ by the convexity of f. Since $\underline{u}(\lambda): [0, \lambda^*] \to [C_0^{2+\alpha}(\bar{\Omega})]^k$ is continuous, it follows from [28] that $\mu_1(\lambda) \to \mu_1(\lambda^*)$. Hence $\lambda^* \leqslant \mu_1(\lambda^*)$ and thus $\lambda^* = \mu_1(\lambda^*)$, since $H_u(\lambda^*, \underline{u}(\lambda^*))$ is singular; (ii) now follows from (H6) and results of [10].

If there exists $u \geqslant \underline{u}(\lambda^*)$ such that $Lu = \lambda^* f(x, u)$,

$$\begin{aligned} L(u - \underline{u}(\lambda^*)) &= \lambda^*[f(x, u) - f(x, \underline{u}(\lambda^*))] \\ &= \lambda^*\left(\int_0^1 f_u(x, \theta u + (1-\theta)\underline{u}(\lambda^*))\, d\theta\right] \cdot (u - \underline{u}(\lambda^*)) \end{aligned} \tag{2.2}$$

The maximum principle implies that $u \gg \underline{u}(\lambda^*)$. (2.2) is equivalent to

$$\begin{aligned} L(u - \underline{u}(\lambda^*)) &= \lambda^* f_u(x, \underline{u}(\lambda^*)) \cdot (u - \underline{u}(\lambda^*)) \\ &\quad + \lambda^*\left\{\left[\int_0^1 f_u(x, \theta u + (1-\theta)\underline{u}(\lambda^*))\, d\theta\right] - f_u(\lambda, \underline{u}(\lambda^*))\right\} \cdot (u - \underline{u}(\lambda^*)). \end{aligned} \tag{2.3}$$

Convexity of f and $u - u(\lambda^*) \gg 0$ imply that the second term of the right hand side of (2.3) is strictly positive, a contradiction to lemma 2.1 of [11], which establishes (i).

To see (iii), we apply the local inversion theory (theorem 2.1) of [1]. The Krein-Rutman theorem implies the existence of a positive linear functional f^* such that $f^* \equiv \lambda^*(L^{-1}f_u(x, \underline{u}(\lambda^*))^* f^*$, where $(L^{-1}f_u(x, \underline{u}(\lambda^*))$ is viewed as a map on, say, $[C_0^0(\bar{\Omega})]^k$ and $(L^{-1}f_u(x, \underline{u}(\lambda^*))^*$ denotes the dual map. It follows from a simple computation that if $h \in [C_0^{2+\alpha}(\bar{\Omega})]^k$ and $[I - \lambda^* L^{-1}f_u(x, \underline{u}(\lambda^*))]w = h$, then $f^*h = 0$. Furthermore, the simplicity of λ^* guarantees that $[C_0^{2+\alpha}(\bar{\Omega})]^k = \langle\phi\rangle \oplus \mathrm{im}(I - \lambda^* L^{-1}f_u(x, u(\lambda^*)))$. Hence if $w \in [C_0^{2+\alpha}(\bar{\Omega})]^k$, w is uniquely expressable as $w = c\phi + r$, where $r \in \mathrm{im}(I - \lambda^* L^{-1}f_u(k, \underline{u}(\lambda^*)))$. Since $\phi \gg 0$ and $f^* \geqslant 0$, there is no loss of generality in assuming $f^*(\phi) = 1$. Thus $f^*(w) = cf^*(\phi) + f^*(r) = c$. Since $w \in \mathrm{im}(I - \lambda^* L^{-1}f_u(x, \underline{u}(\lambda^*)))$ only if $c = 0$, we see that

$$\mathrm{im}(I - \lambda^* L^{-1}f_u(x, u(\lambda^*))) = \{h \in [C_0^{2+\alpha}(\bar{\Omega})]^k : f^*h = 0\}.$$

Furthermore,

$$H_\lambda(\lambda^*, \underline{u}(\lambda^*)) = -L^{-1}f(x, u(\lambda^*)) = -\underline{u}(\lambda^*)/\lambda^*.$$

Thus $f^*(H_\lambda(\lambda^*, \underline{u}(\lambda^*))) \neq 0$ and the conditions of theorem 2.1 of [1] are met. (iii) is now immediate, and, in addition, if $\sigma = 0$, $\lambda'(\sigma) = 0$ and $\lambda''(\sigma) = -f^*(L^{-1}f_{uu}(x, u(\lambda^*))[\phi]^2)/f^*(\underline{u}(\lambda^*))$. This last expression is negative when, for example, f is strictly convex.

COROLLARY 2.4. If $M < \infty$, there is $\delta > 0$ such that if $\lambda \in (\lambda^* - \delta, \lambda^*)$, (0.1) has at least two positive solutions.

Let $0 < \lambda < \lambda^*$ and let $M_\lambda = \sup\{\|u\|_{[C_0^0(\bar{\Omega})]^k} : (\mu, u) \in \Sigma$ for some $\mu \geqslant \lambda\}$. We now have the following global result.

THEOREM 2.5. Suppose that

(i) f is convex
(ii) $M_\lambda < \infty$ for each $\lambda \in (0, \lambda^*)$.

Then (0.1) has at least two solutions in $\mathscr{C}$ for each $\lambda \in (0, \lambda^*)$.

Remark. The proof of this result is essentially the same as that of corollary 2.2 of [18]. However, as we intended to give a several parameter extension of this theorem, we include a brief sketch of the proof.

Proof of theorem 2.5. Let $K = \{u \in [C_0^0(\bar{\Omega})]^k : u \geqslant 0\}$. Define $\Phi(\lambda, u) = \lambda L^{-1}f(x, u)$. Since f is convex, $I - (\partial\Phi/\partial u)(\lambda, \underline{u}(\lambda))$ is invertible by corollary 4.2 of [11]. Then for $\lambda \in [0, \lambda^*)$, there exists $\rho(\lambda) > 0$ such that $i_K(\Phi(\lambda, .), \mathcal{O}_\lambda) = i_K(A_\lambda, \mathcal{O}_\lambda) = +1$, where $\mathcal{O}_\lambda = \{u \in K : \|u - u(\lambda)\| < \rho(\lambda)\}$ and $A_\lambda u \equiv \underline{u}(\lambda) + (\partial\Phi/\partial u)(\lambda, \underline{u}(\lambda) \cdot (u - \underline{u}(\lambda)) \equiv \underline{u}(\lambda) + \lambda L^{-1}f_u(x, \underline{u}(\lambda)) \cdot (u - \underline{u}(\lambda))$. (Here i_K denotes the fixed point index in K; see [3, 17, 18].)

Given $\varepsilon > 0$, (ii) implies that $\mathscr{C}_\varepsilon = \{(\lambda, u) \in \mathscr{C} : \lambda \geqslant \varepsilon\}$ is a bounded set. Hence there exists a bounded set $W \subseteq [\varepsilon, \infty) \times [C_0^0(\bar{\Omega})]^k$ such that $\mathscr{C}_\varepsilon \subseteq W$, W is open in $[\varepsilon, \infty) \times [C_0^0(\bar{\Omega})]^k$, and $w \neq \Phi(\lambda, w)$ for $(\lambda, w) \in \bar{W} - W$.

Let $W_\lambda = \{w : (\lambda, w) \in W\}$. Then $i_K(\Phi(\lambda, .), W_\lambda \cap K)$ is constant for $\lambda \geq \varepsilon$. Since $\Phi(\lambda, .)$ has no fixed points for $\lambda > \lambda^*$, the constant is necessarily 0. If, then, for some $\lambda \in [\varepsilon, \lambda^*)$, $\Phi(\lambda, .)$ does not have a fixed point $u \neq \underline{u}(\lambda)$ with $(\lambda, u) \in \mathscr{C}$, then W_λ can be assumed to be a ball about $\underline{u}(\lambda)$ and $i_K(\Phi(\lambda, .), W_\lambda \cap K) = i_K(\Phi(\lambda, .), \mathcal{O}_\lambda) = 1$, a contradiction.

COROLLARY 2.6. If f is convex and $M_{\lambda_0} < \infty$ for $\lambda_0 \in (0, \lambda^*)$, (0.1) has at least two positive solutions for $\lambda \in [\lambda_0, \lambda^*)$.

We next give an extension of theorem 2.5 to the multiparameter setting (0.6). Our principal tool in this regard is the multidimensional global version of the implicit function theorem, due to Fitzpatrick, Massabo, and Pejsachowicz [19]. This result relies on the concept of complementing maps. For the sake of clarity, we pause briefly to recall portions of this theory which are pertinent to our aims.

Let X be a real Banach space, m a positive integer and $\mathscr{A}$ an open subset of $\mathbb{R}^m \times X$. Suppose $r(\lambda, x) = x - R(\lambda, x)$, for $(\lambda, x) \in \mathscr{A}$, where $R : \mathscr{A} \to X$ is a compact, continuous mapping. A continuous map $g : \mathscr{A} \to \mathbb{R}^m$, which maps bounded subsets of $\mathscr{A}$ into bounded sets, will be called a complement for $r : \mathscr{A} \to X$ provided that the Leray–Schauder degree $\deg_{LS}((g, r), \mathscr{A}, 0)$, is defined and nonzero, where $(g, r) : \mathscr{A} \to \mathbb{R}^m \times X$ is defined by $(g, r)(\lambda, x) = (g(\lambda, x), r(\lambda, x))$. We note that (g, r) is a compact perturbation of the identity on $\mathbb{R}^m \times X$. The following result on complementing maps may be found in [19].

PROPOSITION 2.7. Suppose $\lambda_0 \in \mathbb{R}^m$, $\mathscr{A}_{\lambda_0} = \{x \in X : (\lambda_0, x) \in \mathscr{A}\}$ and $r_{\lambda_0} : \mathscr{A}_{\lambda_0} \to X$ is given by $r_{\lambda_0}(x) = r(\lambda_0, x)$. Then $r : \mathscr{A} \to X$ is complemented by $g : \mathscr{A} \to \mathbb{R}^m$ defined by $g(\lambda, x) = \lambda - \lambda_0$ if and only if $\deg_{LS}(r_{\lambda_0}, \mathscr{A}_{\lambda_0}, 0) \neq 0$.

We may now state the result we shall require.

THEOREM 2.8[19]. Let m, X, $\mathscr{A}$ and r be as in the preceding exposition. Assume that $g : \mathscr{A} \to \mathbb{R}^m$ is continuous. Suppose that $V \subseteq \mathscr{A}$ is open and $g : V \to \mathbb{R}^m$ complements $r : V \to X$. Let $K = ((g, r)|_V)^{-1}(0)$. Then there exists a closed, connected subset $\mathscr{C}$ of $r^{-1}(0)$ whose dimension (see Section 2 of [19]) at each point is at least m, $\mathscr{C} \cap K \neq \varnothing$, and at least one of the following properties holds: (a) $\mathscr{C}$ is unbounded; (b) $\mathscr{C} \cap \partial\mathscr{A} \neq \varnothing$; (c) $\mathscr{C} \cap \{(g, r)^{-1}(0) \setminus K\} \neq \varnothing$.

Now consider (0.6). Let $\mathscr{S} = \{(\lambda, u) \in (\bar{\mathbb{R}}_+)^k \times (C_0^0(\bar{\Omega}))^k : (\lambda, u)$ solves (0.6) and $u \geq 0\}$. For $\tau > 0$, let $M_\tau = \sup\{\|u\|_{[C(\bar{\Omega})]^k} : (\lambda, u) \in \mathscr{S},\ \lambda = (\lambda_1, \ldots, \lambda_k)$, where $\lambda_i \geq 0$ and $\|\lambda\|\mathbb{R}^k \geq \tau\}$.

THEOREM 2.9. Consider (0.6). Suppose f is convex and that $M_\tau < \infty$ for $\tau > 0$. Then for each $\lambda \in \operatorname{int} \Lambda$, there exist at least two solutions of (0.6) in $\mathscr{S}$ with the property that $\mathscr{S}$ has dimension at least k at each of the solutions.

Proof. Let $\bar{\mathscr{C}}$ denote the connected component of $\mathscr{S}$ which contains (0, 0). Observe that $\{(\lambda, \underline{u}(\lambda)) : \lambda \in \operatorname{int}(\bar{\mathbb{R}}^+)^k\}$ is a k-manifold which is necessarily contained in $\bar{\mathscr{C}}$. (That such is the case follows from the convexity of f and the implicit function theorem.) Let $\lambda_0 \subset \operatorname{int} \Lambda$. To see that there exists a solution as required distinct from $\underline{u}(\lambda_0)$, we argue as follows. Let

$0<\tau<\|\lambda_0\|$. Let $T_\tau=\{\lambda\in(\bar{\mathbb{R}}_+)^k:\|\lambda\|\geq\tau\}$. Consider $\bar{\mathscr{C}}_\tau=\{(\lambda,u)\in\bar{\mathscr{C}}:\lambda\in T_\tau\}$. Since $M_\tau<\infty$, $\bar{\mathscr{C}}_\tau$ is compact. Hence, as in the proof of theorem 2.5, we may choose W open in $T_\tau\times[C_0^0(\bar\Omega)]^k$ such that $\bar{\mathscr{C}}_\tau\subseteq W$ and $w\neq\Phi(\lambda,w)$ for $(\lambda,w)\in\bar W-W$. Again as in theorem 2.5, the fixed point index

$$i_K(\Phi(\lambda_0,.),(W_{\lambda_0}\cap K)-\bar{\mathcal{O}}_{\lambda_0})$$

is defined and nonzero. Hence if V is taken to be a bounded open subset of $\mathscr{A}=W\cap\{(\text{int}\,T_\tau)\times[C_0^0(\bar\Omega)]^k\}$ such that $V_{\lambda_0}=W_{\lambda_0}-\overline{B(\underline{u}(\lambda_0);\rho(\lambda_0))}$, proposition 2.7 guarantees that $g:V\to\mathbb{R}^k$ given by $g(\lambda,x)=\lambda-\lambda_0$ complements Φ on V. The result now follows from Theorem 2.8.

3. *A PRIORI* ESTIMATES

In this section we give conditions under which solutions to (0.1) satisfy the *a priori* estimates needed for the application of the results of section 2. All of our earlier assumptions on $f(x,u)$ remain in force. We begin with estimates of the type required in theorem 2.5 and corollary 2.6.

LEMMA 3.1. Suppose (H1) holds and for $\mu=1,\ldots,k$,

$$\lim_{u^\mu\to\infty}\left(\frac{f^\mu(x,u)}{u^\mu}\right)=\infty \tag{3.1}$$

uniformly with respect to $x\in\bar\Omega$ and $u^\nu\geq0$, $\nu\neq\mu$, and

$$\lim_{|u|\to\infty}\left(\frac{|f^\mu(x,u)|}{|u|^\beta}\right)=0 \tag{3.2}$$

for $\beta=(n+1)/(n-1)$. Then for each $\lambda\in(0,\lambda^*)$, we have $M_\lambda=\sup\{\|u\|_{[C(\bar\Omega)]^k}:\mu\geq\lambda,(\mu,u)\in\Sigma\}<\infty$. If in addition $f(x,u)$ is convex, theorem 2.5 applies and (0.1) has at least two solutions in $\mathscr{C}$ for each $\lambda\in(0,\lambda^*)$.

Discussion. This lemma is a slightly modified version of the *a priori* estimates obtained in the proof of theorem 1 of [13]. The arguments in [13] give an extension to systems of the *a priori* bounds obtained for a single equation in [9]. The key point is that for any $\lambda_1\in(0,\lambda^*)$, the hypotheses (3.1) and (3.2) are satisfied by $\lambda f(x,u)$ uniformly for $\lambda\in[\lambda_1,\lambda^*]$. Hence we may choose constants k^ν and C_1^ν, with k^ν as large as desired, so that $\lambda f^\nu(x,u)\geq k^\nu u^\nu-C_1^\nu$, and for any $\varepsilon>0$, we can choose $C_2^\nu(\varepsilon)$ so that $\lambda f^\nu(x,u)\leq\varepsilon|u|^\beta+C_2^\nu(\varepsilon)$, for all $\lambda\in[\lambda_1\lambda^*]$. An examination of the proof of theorem 1 of [13] or theorem 2.1 of [9] shows that solutions of (0.1) must satisfy an *a priori* bound depending only on k^ν, C_1^ν, $C_2^\nu(\varepsilon)$ for $\nu=1,\ldots,k$ and on constants depending only on Ω, such as Sovolev embedding constants. Since these constants may be chosen uniformly in λ, the result follows.

If we impose some additional conditions on the domain Ω and the system (0.1), we can weaken the growth condition (3.1).

LEMMA 3.2. Suppose that for $\mu=1,\ldots,k$, we have $L^\mu=-\Delta$. Suppose that $f=f(u)$, (H1) and (H3) hold, and that at each point of $\partial\Omega$ all the sectional curvatures of $\partial\Omega$ are strictly positive. If $f(u)$ satisfies (3.1) and

$$\lim_{|u|\to\infty}\left(\frac{|f(u)|}{|u|^{\gamma}}\right)=0, \tag{3.3}$$

where $\gamma < n/(n-2)$ for $n \geqslant 3$, $\gamma < \infty$ for $n = 2$, then the conclusions of lemma 3.1 hold.

Discussion. Again, the lemma follows directly from the arguments used in theorem 2 of [13], which extend to systems the estimates of [25] for a single equation. As long as $\lambda \in [\lambda_1, \lambda^*]$ with $\lambda_1 > 0$, conditions (3.1) and (3.3) will be satisfied by $\lambda f(u)$ uniformly in λ, so again the *a priori* estimates can be made uniformly in λ.

Remarks. In the case of a single equation $-\Delta u = f(u)$, there exists some *a priori* estimates that permit the weaker growth condition

$$\lim_{|u|\to\beta}\frac{|f(u)|}{|u|^{\beta}}=0$$

with $\beta < (n+2)/(n-2)$ for $n \geqslant 3$; see [18]. However, those estimates depend in part on the existence of a primitive or antiderivative for $f(u)$. In the case of systems the existence of an antiderivative for f, i.e. the requirement that f is a gradient, imposes severe restrictions on the form of the nonlinearity. This point and other differences between the scalar and system cases are discussed in [13].

Lemmas 3.1 and 3.2 both require condition (3.1), which implies a type of superlinearity on f. Another possibility is that f is asymptotically linear, that is, there exists a matrix $f_u(x, \infty)$ such that

$$\lim_{|w|\to\infty} f_u(x, w) = f_u(x, \infty). \tag{3.4}$$

uniformly for $x \in \bar{\Omega}$. We will assume that $f_u(x, \infty)$ is nonnegative irreducible, with positivity somewhere on the diagonal. In our analysis we will generally follow [1]. The structure of our problem is more specialized than those considered in [1], but we cannot apply the results of [1] directly since they require (in our situation) the strong positivity of $\lambda L^{-1}f_u(x, u)$ and/or $\lambda L^{-1}f_u(x, \infty)$. However, we can use some of the results of [10] in place of strong positivity. Since $f_u(x, \infty)$ is nonnegative irreducible with positivity on the diagonal, it follows from [10, theorem 3.4], that the problem

$$\begin{aligned} Lw &= \mu f_u(x,\infty)w \quad \text{in } \Omega \\ w &= 0 \quad\quad\quad\quad\quad\ \text{on } \partial\Omega \end{aligned} \tag{3.5}$$

has a unique eigenvalue with positive eigenfunction. We denote this eigenvalue by λ_∞. Our analysis will be divided into two parts. First, we show that $\lambda_\infty \leqslant \lambda^*$ and that if $\lambda_\infty < \lambda^*$ then all solutions of (0.1) are uniformly bounded for $\lambda \in [\lambda_0, \lambda^*]$ for any $\lambda_0 \in (\lambda_\infty, \lambda^*)$. Second, we obtain conditions under which $\lambda_\infty < \lambda^*$. In the first part we follow [1] rather closely; in the second part the analysis requires more modifications.

To begin our analysis, we observe that (3.4) implies

$$\lim_{\|u\|\to\infty}(\|L^{-1}f(x,u) - L^{-1}f_u(x,\infty)u\|/\|u\|) = 0. \tag{3.6}$$

where the norms are taken in $[C_0^0(\bar{\Omega})]^k$. This follows as in [1, lemma 7.4]. Since f is convex, corollary 4.2 of [11] implies that $\lambda^* < \infty$. Since Σ contains an unbounded component by proposition 2.1, we choose $(\lambda_j, u_j) \in \Sigma$ such that $\|u_j\| \to \infty$. $\{\lambda_j\} \subseteq [0, \lambda^*]$ so we may assume $\lambda_j \to \mu \in [0, \lambda^*]$ as $j \to \infty$. The convexity of f also implies $f_u(x, u) < f_u(x, \infty)$ componentwise, so that for $u \geqslant 0$,

$$\begin{aligned} 0 \leqslant f(x, u) &= f(x, 0) + \left(\int_0^1 f_u(x, tu)\, \mathrm{d}t\right) u \\ &\leqslant f(x, 0) + f_u(x, \infty) u. \end{aligned} \tag{3.7}$$

We then have

$$\begin{aligned} &(u_j/\|u_j\|) - \mu L^{-1} f_u(x, \infty)(u_j/\|u_j\|) \\ &\quad = \mu[L^{-1} f(x, u_j) - L^{-1} f_u(x, \infty) u_j]/\|u_j\| \\ &\qquad + (\lambda_j - \mu) L^{-1} f(x, u_j)/\|u_j\| \end{aligned} \tag{3.8}$$

It follows from (3.6), (3.7), and the boundedness of L^{-1} on $[C_0^0(\bar{\Omega})]^k$ that the right side of (3.8) tends to zero as $j \to \infty$. Also, since $\|(u_j/\|u_j\|)\| = 1$ and $L^{-1} f_u(x, \infty)$ is compact on $[C_0^0(\bar{\Omega})]^k$ we may assume $(u_j/\|u_j\|) \to h$ in $[C_0^0(\bar{\Omega})]^k$ and hence by (3.8) in $[C_0^{1+\alpha}(\bar{\Omega})]^k$, with $\|h\|_{[C(\bar{\Omega})]^k} = 1$ and $h \geqslant 0$. Thus letting $j \to \infty$ in (3.8) yields $h - \mu L^{-1} f_u(x, \infty) h = 0$ so that h and μ satisfy (3.5). Since $h \geqslant 0$, $h \not\equiv 0$, we have $\mu = \lambda_\infty$. Thus we see that $\lambda_\infty \in [0, \lambda^*]$. Suppose now that $\lambda_\infty \notin [\lambda_1, \lambda_2]$, and let $\Gamma = \{(\lambda, u) \in \Sigma : \lambda \in [\lambda_1, \lambda_2]\}$. If Γ is unbounded, we can choose a sequence $(\mu_j, u_j) \in \Gamma$ with $\|u_j\| \to \infty$. But in that case we can argue as above and conclude $\lambda_\infty \in [\lambda_1, \lambda_2]$, a contradiction, thus, Γ must be bounded. Combining the above arguments with corollary 2.6, we have the following:

LEMMA 3.3. If $\lambda_\infty < \lambda^*$, (0.1) has at least two positive solutions on $(\lambda_\infty, \lambda^*)$.

Proof. By the discussion above, if $\lambda_\infty < \lambda^*$ then $\{(\lambda, u) \in \Sigma : \lambda \in [\lambda_0, \lambda^*]\}$ is bounded for all $\lambda_0 \in (\lambda_\infty, \lambda^*)$; that is, $M_{\lambda_0} < \infty$. By corollary 2.6, (0.1) has at least two solutions for $\lambda \in [\lambda_0, \lambda^*)$. Since $\lambda_0 \in (\lambda_\infty, \lambda^*)$ was arbitrary, (0.1) must have at least two solutions for $\lambda \in (\lambda_\infty, \lambda^*)$.

Remark. The arguments preceding lemma 2.3 are essentially those used in proving lemmas 5.2–5.5 of [1].

We now turn to the problem of finding conditions which imply $\lambda_\infty < \lambda^*$. The following lemma will be useful:

LEMMA 3.4. Let $M(x)$ be a $k \times k$ matrix of nonnegative continuous functions with $M(x_0)$ irreducible and having positive entries on the diagonal for some $x_0 \in \Omega$. Then the problem

$$\begin{aligned} L\phi &= \mu_1 M \phi && \text{in } \Omega \\ \phi &= 0 && \text{on } \partial\Omega \end{aligned} \tag{3.9}$$

has a unique eigenvalue $\mu_1(M) > 0$ with eigenfunction $\phi \gg 0$. Further, (3.9) may be rewritten as $\phi = \mu_1 L^{-1}M\phi$ with $L^{-1}M$ interpreted as an operator on $[C_0^0(\bar{\Omega})]^k$, and there exists a nontrivial nonnegative linear functional ϕ^* on $([C_0^0(\Omega)]^k)^*$ such that

$$\phi^* = \mu_1(M)(L^{-1}M)^*\phi^* \tag{3.10}$$

Finally, ϕ^* has the property that if $b \in [C_0^1(\bar{\Omega})]^k$ with $b \gg 0$, then $\langle \phi^*, b\rangle > 0$.

Proof. The first part of the lemma, that is, the existence and properties of μ_1 and ϕ, follows from theorems 2.3 and 3.4 and lemma 3.1 of [10]. The argument in [10] is based partly on the fact that $L^{-1}M$ viewed as an operator on $[C_0^0(\bar{\Omega})]^k$ is compact and positive. Thus the Krein–Rutman theorem applies, which in turn gives the existence of ϕ^*. To see that $\langle \phi^*, b\rangle > 0$ for $b \gg 0$, we argue as follows: since ϕ^* is nonnegative and nontrivial, there exists $a \in [C_0^0(\bar{\Omega})]^k$ with $\langle \phi^*, a\rangle > 0$. We can approximate a as closely as desired with respect to the $[C_0^0(\bar{\Omega})]^k$ norm with $\tilde{a} \in [C_0^1(\bar{\Omega})]^k$, so $\langle \phi^*, \tilde{a}\rangle > 0$. We have $\langle \phi^*, b\rangle \geq 0$ since $b \gg 0$. Suppose $\langle \phi^*, b\rangle = 0$. Then $\langle \phi^*, b - \varepsilon\tilde{a}\rangle = -\varepsilon\langle \phi^*, \tilde{a}\rangle < 0$ for any $\varepsilon > 0$. But for ε sufficiently small, $b - \varepsilon\tilde{a} \gg 0$ so $b - \varepsilon\tilde{a} \geq 0$; hence $\langle \phi^*, b - \varepsilon\tilde{a}\rangle \geq 0$, a contradiction. Thus we must have $\langle \phi^*, b\rangle > 0$.

Remark. Lemma 3.4 gives precisely the results needed to replace the hypotheses involving strong positivity of operators used in [1]. The existence of a smallest eigenvalue $\mu_1(M) > 0$ with nontrivial nonnegative eigenfunction does not require irreducibility on M; see [10, theorem 2.3, corollary 2.4].

Let $e = L^{-1}\mathbb{1}$ where $\mathbb{1}$ is the column vector with all entries equal to 1. Then $e \gg 0$ and we have

LEMMA 3.5. Suppose that $f(x, u)$ is asymptotically linear and there exists $\xi_0 \in \mathbb{R}$ such that if $\xi \in (\mathbb{R}^+)^k$ and $|\xi| > \xi_0$,

$$f(x, \xi) - f_u(x, \xi)\xi \leq -g(x) \in [C_0^0(\bar{\Omega})]^k, \tag{3.11}$$

where $g \geq 0$ and for some $x_0 \in \Omega$, $g^{\mu}(x_0) > 0$ for $\mu = 1, \ldots, k$. Then there exists $\alpha \in \mathbb{R}$, $\alpha > 0$ such that for $u \geq \alpha e$,

$$L^{-1}(f(x, u) - f_u(x, u)u) \ll 0. \tag{3.12}$$

Discussion. This is essentially lemma 7.8 of [1]; the proof in our situation requires only minor modifications.

LEMMA 3.6. Suppose that there exists $\hat{e} \gg 0$ such that if $(\lambda, u) \in \Sigma$ with $\lambda > 0$, $u \geq \hat{e}$, then (3.12) holds. Then for $(\lambda, u) \in \Sigma$ with $\lambda > 0$ and $u \geq \hat{e}$, $\mu_1(f_u(x, u)) < \lambda$.

Proof. By lemma 3.4, there exists a nonnegative linear functional ϕ^* on $[C_0^0(\bar{\Omega})]^k$ satisfying $\phi^* = \mu_1(f_u(x, u))[L^{-1}f_u(x, u)]^*\phi^*$ such that for $b \gg 0$, $\langle \phi^*, b\rangle > 0$. Since $(\lambda, \mu) \in \Sigma$, (3.12) implies that if $u > \hat{e}$,

$$\left(\frac{1}{\lambda}\right)u - L^{-1}f_u(x, u)u \ll 0. \tag{3.13}$$

Applying ϕ^* to (3.13) yields

$$\left(\frac{1}{\lambda}\right)\langle\phi^*, u\rangle - \langle\phi^*, L^{-1}f_u(x, u)u\rangle < 0$$

so that

$$\left[\left(\frac{1}{\lambda}\right) - \frac{1}{\mu_1(f_u(x, u))}\right]\langle\phi^*, u\rangle = \left(\frac{1}{\lambda}\right)\langle\phi^*, u\rangle - \langle[L^{-1}f_u(x, u)]^*\phi^*, u\rangle < 0.$$

Thus, since $u \geqslant \hat{e} \gg 0$ we have $\langle\phi^*, u\rangle < 0$ so that $(1/\lambda) - (1/\mu_1(f_u(x, u))) < 0$ as desired.

The following is essentially theorem 5.10 of [1]:

LEMMA 3.7. Suppose that $f(x, u)$ is asymptotically linear and there exists $\gamma > 0$ such that if $u \geqslant \gamma e$, $\lambda > 0$, and $(\lambda, u) \in \Sigma$, then $\lambda > \mu_1(f_u(x, u))$. Then $\lambda_\infty < \lambda^*$.

Discussion. The proof is essentially the same as that of theorem 5.10 of [1], which in turn depends on lemmas 4.5 and 4.6 of the same article. Both of these results require strong positivity of the operators analogous to $L^{-1}f_u(x, u)$; however, all that is needed for the proofs is the existence of an eigenfunction $\phi \gg 0$ for $L^{-1}f_u(x, u)$, which follows from lemma 3.4. Hence the analysis carries over to our situation.

We can summarize the above results in the following lemma:

LEMMA 3.8. Suppose that $f(x, u)$ is asymptotically linear and satisfies hypothesis (3.11). Then $\lambda_\infty < \lambda^*$ and (0.1) has at least two solutions for $\lambda \in (\lambda_\infty, \lambda^*)$.

So far we have considered only *a priori* estimates on all solutions of (0.1) for some range of λ. However, theorem 2.3 and corollary 2.4 require only that the minimal solutions u_λ satisfy an *a priori* bound. In some cases such bounds can be obtained when bounds for all solutions cannot. In what follows we show how the methods of Crandall and Rabinowitz [16] can be adapted to systems to yield such results. To apply the methods of [16], we must assume that the operators L^α on the diagonal of L have self-adjoint form; and we must also compare the eigenvalues for certain symmetric problems to those from the original, nonsymmetric problem. We will need the following lemma.

LEMMA 3.9. Suppose that $M(x)$ and $Q(x)$ are $k \times k$ matrices of class C^1, having the form

$$M = \begin{pmatrix} M^1 & & 0 \\ & \ddots & \\ 0 & & M^{N_1} \end{pmatrix} \qquad Q = \begin{pmatrix} Q^1 & & 0 \\ & \ddots & \\ 0 & & Q^{N_2} \end{pmatrix}$$

with each block M^α, Q^α nonnegative in Ω, and irreducible with positivity on the diagonal at some point in Ω. Suppose that the entries in M and Q satisfy $Q_{\alpha\beta} \leqslant M_{\alpha\beta}$, $\alpha, \beta = 1, \ldots, k$. If $\mu_1(M)$ is the first eigenvalue for (3.9) and $\lambda < \mu_1(M)$, then the problem

$$\begin{aligned} L\psi - \lambda Q(x)\psi &= \sigma\psi \quad \text{in } \Omega \\ \psi &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.14}$$

has a positive smallest eigenvalue; the eigenfunction ψ is nonnegative and nontrivial.

Proof. The existence of a positive smallest eigenvalue $\mu_1(M)$ for the problem $Lu = \mu Mu$ in Ω, $u = 0$ on $\partial\Omega$, follows from [10, theorem 2.3 and corollary 2.4]; similarly for $\mu_1(Q)$. By lemma 2.1 and corollary 2.3 of [11], we see that if $\lambda^*(M)$ is the least upper bound of the values of λ for which $Lu = \lambda(\mathbf{1} + Mu)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω has a positive solution, then $\lambda^*(M) = \mu_1(M)$. Similarly, $\lambda^*(Q) = \mu_1(Q)$, and since $Q_{\alpha\beta} \leqslant M_{\alpha\beta}$, it follows by corollary 3.4 of [11] that $\mu_1(Q) = \lambda^*(Q) \geqslant \lambda^*(M) = \mu_1(M)$. Hence if $\lambda < \mu_1(M)$, then $\lambda < \mu_1(Q)$. If $N_2 > 1$, we can characterize $\mu_1(Q)$ by observing that the system $Lu = \mu Qu$ decouples into N_2 systems of the form $\bar{L}^\gamma u = \mu Q^\gamma u$, where $\bar{L}^\gamma$ represents the part of the matrix L corresponding to the block Q^γ. We may apply lemma 3.4 to the problem $\bar{L}^\gamma u = \mu Q^\gamma u$ and assert that for each γ, there is a positive smallest eigenvalue $\mu_1(Q^\gamma)$ with eigenfunction $\phi^\gamma \gg 0$. Hence if μ is an eigenvalue of $Lu = \mu Qu$, then it must also be an eigenvalue for $\bar{L}^\gamma u = \mu Q^\gamma u$ for some γ; similarly, if we take $u = \phi^\gamma$ in the components corresponding to Q^γ and zero in all other components, we obtain an eigenfunction for $Lu = \mu Mu$ with eigenvalue $\mu_1(Q^\gamma)$. Thus, $\mu_1(Q) = \min\{\mu_1(Q^1), \ldots, \mu_1(Q^N2)\}$, so $\lambda < \mu_1(Q^\gamma)$ for all γ. We can rewrite (3.14) as

$$(\bar{L}^\beta - \lambda Q^\beta)\psi^\beta = \sigma\psi^\beta \quad \text{in } \Omega, \qquad \psi^\beta = 0 \quad \text{on } \partial\Omega, \qquad \beta = 1, \ldots, N_2. \tag{3.15}$$

Again, if each of the problems in (3.15) has a positive first eigenvalue σ_β, then the first eigenvalue for (3.15) is given by $\sigma_0 = \min\{\sigma_1, \ldots, \sigma_{N_2}\} > 0$. Since $\lambda < \mu_1(Q^\beta)$ for each β, (3.16) may be rewritten as

$$\psi^\beta = \sigma(\bar{L}^\beta - \lambda Q^\beta)^{-1}\psi^\beta \tag{3.16}$$

where $(\bar{L}^\beta - \lambda Q^\beta)^{-1}$ is a positive, compact operator. (This follows from lemma 2.1 of [11].) Thus, if $\bar{L}^\beta - \lambda Q^\beta$ has a positive spectral radius, (3.16) has a positive first eigenvalue by the Krein–Rutman theorem, so that (3.14) does, also. Thus it remains only to show that the spectral radius of $(\bar{L}^\beta - \lambda Q^\beta)^{-1}$ is positive. The problem $[\bar{L}^\beta - \mu_1(Q^\beta)Q^\beta]\phi = 0$ in Ω, $\phi = 0$ on $\partial\Omega$ has a solution $\phi \gg 0$, so $(\bar{L}^\beta - \lambda Q^\beta)\phi = (\mu_1(Q^\beta) - \lambda)Q^\beta\phi$. Letting $h = (\mu_1(Q^\beta) - \lambda)Q^\beta\phi$, we have $h \geqslant 0$ and $h(x_0) > 0$ for some $x_0 \in \Omega$ via the hypotheses on Q^β. Thus $\|h\| > 0$ in $[C_0^0(\bar{\Omega})]^{k_\beta}$ (where Q^β is a $k_\beta \times k_\beta$ matrix). Since $\phi \gg 0$, it follows that for some $\varepsilon > 0$

$$0 \leqslant \varepsilon h \leqslant \phi = (\bar{L}^\beta - \lambda Q^\beta)^{-1}h. \tag{3.17}$$

Applying $(\bar{L}^\beta - \lambda Q^\beta)^{-1}$ to (3.17) and using (3.17) again, we obtain

$$0 \leqslant \varepsilon^2 h \leqslant \varepsilon(\bar{L}^\beta - \lambda Q^\beta)^{-1}h \leqslant (\bar{L}^\beta - \lambda Q^\beta)^{-2}h, \tag{3.18}$$

so proceeding in this way we obtain by induction $0 \leqslant \varepsilon^m h \leqslant (\bar{L}^\beta - \lambda Q^\beta)^{-m}h$ so that $\varepsilon^m|h| \leqslant |(\bar{L}^\beta - \lambda Q^\beta)^{-m}h|$, and hence in $[C_0^0(\bar{\Omega})]^{k_\beta}$ we have $\varepsilon^m\|h\| \leqslant \|(\bar{L}^\beta - \lambda Q^\beta)^{-m}\|\,\|h\|$. Since $\|h\| > 0$ we have $\varepsilon^m \leqslant \|(\bar{L}^\beta - \lambda Q^\beta)^{-m}\|$ so $0 < \varepsilon \leqslant \|(\bar{L}^\beta - \lambda Q^\beta)^{-m}\|^{1/m}$. Hence $(\bar{L}^\beta - \lambda Q^\beta)^{-1}$ has a positive spectral radius, which is the result we need to complete the proof.

We can now describe how the methods of [16] may be extended to systems. We must assume that the operators L^α have variational form, that is, for $\gamma = 1, \ldots, k$,

$$L^\gamma w = -\sum_{i,j=1}^{n} (a_{ij}^\gamma(x)w_{x_i})_{x_j} + c^\gamma(x)w \tag{3.19}$$

with a_{ij}^{γ} of class $C^{1+\alpha}$, and $a_{ij}^{\gamma} = a_{ji}^{\gamma}$, We must also assume that $f(x, u)$ is convex and of class C^2. Finally, suppose there exists a $k \times k$ matrix $G(x, u)$ of class C^1 such that the entries of G satisfy

$$0 \leqslant G_{\alpha\beta}(x, u) \leqslant \frac{\partial f^{\alpha}}{\partial u^{\beta}}(x, u) \quad \text{for } u \geqslant 0 \tag{3.20}$$

where G has the form

$$G = \begin{pmatrix} G_1 & & 0 \\ & \ddots & \\ 0 & & G_N \end{pmatrix}$$

with each of the matrices $G_i(x, u)$ symmetric, and irreducible with positivity on the diagonal at some point of Ω provided $u \geqslant 0$.

The idea of the analysis is to use the variational characterization of the first eigenvalue σ_0 for $(L - \lambda G(x, \underline{u}_\lambda))w = \sigma w$ together with the conclusion of lemma 3.9, which implies $\sigma_0 > 0$, to obtain an identity which can yield estimates on $\underline{u}_\lambda$. Under the above hypotheses, we have

LEMMA 3.10. Suppose that the operators L^{γ} satisfy (3.19) and that (3.20) holds. If $\phi(u) = \operatorname{col}(\phi^1(u), \ldots, \phi^k(u))$ satisfies $\phi(0) = 0$ and is of class C^1, then for $\lambda \in [0, \lambda^*)$ and $u = \underline{u}_\lambda$ we have

$$\sum_{\gamma=1}^{k} \int_{\Omega} \left[\sum_{\alpha,\beta=1}^{k} \sum_{i,j=1}^{n} a_{ij}^{\gamma}(x) \phi_{u^{\alpha}}^{\gamma}(u) \phi_{u^{\beta}}^{\gamma}(u) u_{x_i}^{\alpha} u_{x_j}^{\beta} + c^{\gamma}(x) \phi^{\gamma}(u) \phi^{\gamma}(u) \right] dx$$
$$\geqslant \lambda \int_{\Omega} \left[\sum_{\alpha,\beta=1}^{k} G^{\alpha\beta}(x, u) \phi^{\alpha}(u) \phi^{\beta}(u) \right] dx. \tag{3.21}$$

Proof. By corollary 4.2 of [11], $\lambda < \mu_1(f_u(x, \underline{u}_\lambda))$ for $\lambda \in [0, \lambda^*)$ when $f(x, u)$ is convex. Thus, by (3.20) and lemma 3.9, we have $\sigma > 0$ where σ is the first eigenvalue of

$$[L - \lambda G(x, \underline{u}_\lambda)]w = \sigma w \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega. \tag{3.22}$$

The hypotheses on L and G imply that if $G(x, \underline{u}_\lambda)$ is viewed as a function of x, then (3.22) has a variational structure; so σ can be found by minimizing the expression

$$E = \int_{\Omega} \left(\sum_{\gamma=1}^{k} \left[\sum_{i,j=1}^{n} a_{ij}^{\gamma} w_{x_i}^{\gamma} w_{x_j}^{\gamma} + c^{\gamma} w^{\gamma} w^{\gamma} \right] - \lambda \sum_{\alpha,\beta=1}^{k} G^{\alpha\beta}(x, \underline{u}_\lambda) w^{\alpha} w^{\beta} \right) dx$$

over $w \in [W_0^{1,2}(\Omega)]^k$ with $\|w\| = 1$. Since $\sigma > 0$, we have $E \geqslant 0$ for any $w \in [W_0^{1,2}(\Omega)]^k$. Letting $w = \phi(\underline{u}_\lambda)$ yields (3.21).

We can obtain another identity which, when combined with (3.21), may yield *a priori* bounds on $\underline{u}_\lambda$ by multiplying (0.1) by an appropriate vector and integrating by parts. Suppose that $\psi(u) = \operatorname{col}(\psi^1(u), \ldots, \psi^k(u))$ is of class C^1 with $\psi(0) = 0$. Taking the scalar product of (0.1) with $\psi(u)$ and integrating by parts using the divergence theorem yields (since $a_{ij}^{\beta} = a_{ji}^{\beta}$)

$$\sum_{\beta=1}^{k} \int_{\Omega} \left[\sum_{\alpha=1}^{k} \sum_{i,j=1}^{n} a_{ij}^{\beta}(x) \psi_{u^{\alpha}}^{\beta}(u) u_{x_i}^{\alpha} u_{x_j}^{\beta} + c^{\beta} u^{\beta} \psi^{\beta}(u) \right] dx$$
$$= \lambda \int_{\Omega} \left[\sum_{\beta=1}^{k} \psi^{\beta}(u) f^{\beta}(x, u) \right] dx \tag{3.23}$$

If we have $u = \underline{u}_\lambda$, we can combine (3.21) and (3.23) to get an inequality free from derivatives of u provided we have

$$\sum_{\alpha,\beta=1}^{k} \sum_{i,j=1}^{n} a_{ij}^{\beta} \psi_{u^\alpha}^{\beta} u_{x_i}^{\alpha} u_{x_j}^{\beta} \geq \sum_{\alpha,\beta,\gamma=1}^{k} \sum_{i,j=1}^{n} a_{ij}^{\gamma} \phi_{u^\alpha}^{\gamma} \phi_{u^\beta}^{\gamma} u_{x_i}^{\alpha} u_{x_j}^{\beta}. \tag{3.24}$$

In general we have little information on $u_{x_i}^{\alpha}$; so we will want to choose ψ and ϕ so that for $\xi \in \mathbb{R}^{kn}$, $\xi = (\xi_i^\alpha)$ $\alpha = 1, \ldots, k$; $i = 1, \ldots, n$, we have

$$\sum_{\alpha,\beta=1}^{k} \sum_{i,j=1}^{n} a_{ij}^{\beta} \psi_{u^\alpha}^{\beta} \xi_i^{\alpha} \xi_j^{\beta} \geq \sum_{\alpha,\beta,\gamma=1}^{k} \sum_{i,j=1}^{n} a_{ij}^{\gamma} \phi_{u^\alpha}^{\gamma} \phi_{u^\beta}^{\gamma} \xi_i^{\alpha} \xi_j^{\beta}. \tag{3.25}$$

Suppose $\psi^\alpha(u) = \psi^\alpha(u^\alpha)$, $\phi^\alpha(u) = \phi^\alpha(u^\alpha)$, $\alpha = 1, \ldots, k$; then $\psi_{u^\beta}^{\alpha} = 0$ if $\beta \neq \alpha$; $\psi_{u^\alpha}^{\alpha} = (\psi^\alpha)'$, and similarly for ϕ^α. Thus (3.25) becomes

$$\sum_{\alpha=1}^{k} (\psi^\alpha)' \left(\sum_{i,j=1}^{n} a_{ij}^{\alpha} \xi_i^{\alpha} \xi_j^{\alpha} \right) \geq \sum_{\alpha=1}^{k} [(\phi^\alpha)']^2 \left(\sum_{i,j=1}^{n} a_{ij}^{\alpha} \xi_i^{\alpha} \xi_j^{\alpha} \right). \tag{3.26}$$

Thus, (3.24) holds in this case provided

$$(\psi^\alpha)' \geq [(\phi^\alpha)']^2, \qquad \alpha = 1, \ldots, k, \tag{3.27}$$

which is precisely the vector analog of a condition used for a single equation in [16].

Suppose now that (3.24) holds. Letting $u = u_\lambda$ and combining (3.21) and (3.23) via (3.24) yields

$$\lambda \int_\Omega \sum_{\beta=1}^{k} \psi^\beta f^\beta \, dx \geq \lambda \int_\Omega \sum_{\alpha,\beta=1}^{k} G^{\alpha\beta} \phi^\alpha \phi^\beta \, dx + \int_\Omega \sum_{\beta=1}^{k} c^\beta \psi^\beta u^\beta \, dx - \int_\Omega \sum_{\beta=1}^{k} c^\beta (\phi^\beta)^2 \, dx. \tag{3.28}$$

Inequality (3.28) is fairly general; to obtain estimates from it we must put conditions on f and then choose ψ and ϕ carefully. For a single equation, various special cases are treated in [16]; we will examine the system analogues to some of those.

Suppose that $f^\alpha(x, u)$ has the form

$$f^\alpha(x, u) = g^\alpha(x)(u^\alpha)^{m_\alpha} + h^\alpha(x, u) \tag{3.29}$$

with $m_\alpha > 1$, $g^\alpha(x) > 0$ on $\bar{\Omega}$, and $f(x, u)$ satisfies the conditions following (3.19). Let $m_0^* = \min_\alpha(m_\alpha)$; then $m_0^* > 1$. We will assume that for some $\delta > 0$ we have for all α

$$|h^\alpha(x, u)| < C(1 + |u|^{m_0^* - \delta}) \tag{3.30}$$

and for all α, β

$$\frac{\partial h^\alpha}{\partial u^\beta} \geq 0. \tag{3.31}$$

Let $G^{\alpha\alpha} = m_\alpha g^\alpha(x)(u^\alpha)^{m_\alpha - 1}$ and $G^{\alpha\beta} = 0$, $\alpha \neq \beta$. Then (3.20) holds. Let $\phi^\alpha(u) = (u^\alpha)^{j_\alpha}$ and $\psi^\alpha(u) = [j_\alpha^2/(2j_\alpha - 1)](u^\alpha)^{2j_\alpha - 1}$ so that (3.27) holds. Then (3.28) will be satisfied for $u = \underline{u}_\lambda$; so when $u = \underline{u}_\lambda$, we have

$$\lambda \int_\Omega \sum_{\alpha=1}^{k} \left[\frac{j_\alpha^2}{(2j_\alpha - 1)}\right] g^\alpha (u^\alpha)^{(2j_\alpha + m_\alpha - 1)}\, dx$$

$$\geqslant \lambda \int_\Omega \sum_{\alpha=1}^{k} m_\alpha g^\alpha (u^\alpha)^{(2j_\alpha + m_\alpha - 1)}\, dx$$

$$- \int_\Omega \sum_{\alpha=1}^{k} c^\alpha (u^\alpha)^{2j_\alpha}\, dx$$

$$+ \int_\Omega \sum_{\alpha=1}^{k} \left[\frac{j_\alpha^2}{2j_\alpha - 1}\right] [c^\alpha u^2 - \lambda h^\alpha](u^\alpha)^{2j_\alpha - 1}\, dx \tag{3.32}$$

If we set $W = \sum_{\alpha=1}^{k} g^\alpha (u^\alpha)^{(2j_\alpha + m_\alpha - 1)}$, we see that (3.30) and the fact that $m_\alpha > 1$ for all α imply that the first term on each side of (3.32) is of order 1 in W and the remaining terms are of lower order. If $m_\alpha > [j_\alpha^2/2j_\alpha - 1]$ for all α and we restrict λ to, say, $[\lambda^*/2, \lambda^*)$, then (3.32) implies a bound on W, uniform in $\lambda \in [\lambda^*/2, \lambda^*)$. As discussed in [16], the relation between m_α and j_α can be satisfied if $m_\alpha - (m_\alpha^2 - m_\alpha)^{1/2} < j_\alpha < m_\alpha + (m_\alpha^2 - m_\alpha)^{1/2}$ or equivalently $2j_\alpha + m_\alpha - 1 < m_\alpha[2 + 2\sqrt{\gamma_\alpha} + \gamma_\alpha]$, where $\gamma_\alpha = 1 - (1/m_\alpha)$. The bound on W implies a bound on $\|u^\alpha\|$ in $L^p(\Omega)$ with $p = 2j_\alpha + m_\alpha - 1$, so for any $m_0 < m_0^*$ we can bound $\|u\|$ in $[L^{p_0}(\Omega)]^k$, where $p_0 = m_0[2 + 2\sqrt{\gamma_0} + \gamma_0]$ with $\gamma_0 = 1 - (1/m_0)$. Let $M_0 = \max_\alpha(m_\alpha)$. By (3.30) we can bound $\|f\|$ in $[L^{p_0/M_0}(\Omega)]^k$, since $|f| \leqslant K(1 + |u|^{M_0})$. We may now use the usual bootstrap process as discussed in [16]: we obtain a bound for u in $[W^{2,p_0/M_0}(\Omega)]^k$ via (0.1) and the standard elliptic *a priori* estimates; the Sobolev imbedding theorem then implies a bound for u in $[L^{p_1}(\Omega)]^k$ for any

$$p_1 < \frac{n(p_0/M_0)}{[n - 2(p_0/M_0)]},$$

which then yields a bound for f in $[L^{p_1/M_0}(\Omega)]^k$, and so on. This process eventually yields a bound for u in $[W^{2,p}(\Omega)]^k$ with $2p > n$ so that $W^{2,p}(\Omega)$ imbeds in $C_0(\bar{\Omega})$ provided

$$2\left(\frac{p_0}{M_0}\right) > n\left[1 - \left(\frac{1}{M_0}\right)\right]; \tag{3.33}$$

see lemma 1.17 of [16]. The imbedding then gives a bound for $u = \underline{u}$ in $[C^0(\bar{\Omega})]^k$ if $\lambda \in [\lambda^*/2, \lambda^*)$. For $\lambda \in [0, \lambda^*/2)$, $0 \leqslant \underline{u}_\lambda \leqslant \sup_\Omega \underline{u}_{\lambda^*/2}$, so it follows that

$$M = \sup\{\|\underline{u}_\lambda\|_{[C_0^0(\bar{\Omega})]^k} : \lambda \in [0, \lambda^*)\} < \infty.$$

Inequality (3.33) combined with the bound for possible values of p_0 gives the inequality

$$n < 2m_0 \frac{[2 + 2\sqrt{\gamma_0} + \gamma_0]}{(M_0 - 1)} \tag{3.34}$$

(recall $\gamma_0 = 1 - (1/m_0)$), which reduces to the bound on n obtained in [16] for a single equation with similar nonlinearity provided $m_\alpha = m_0^* = M_0$ for all α.

We summarize the above analysis in the following:

LEMMA 3.11. Suppose that L satisfies (3.19), that f satisfies (H1), (H2), (H5) and (H6), is convex, and has the form (3.29), and that (3.34) holds. Then $M = \sup\{\|\underline{u}_\lambda\|_{[C_0^0(\bar{\Omega})]^k} : \lambda \in [0, \lambda^*)\} < \infty$, so by corollary 2.4 there exists $\delta > 0$ such that for $\lambda \in (\lambda^* - \delta, \lambda^*)$, (0.1) has at least two positive solutions.

Remark. Condition (3.34) is in a sense a growth condition on f. The condition can be satisfied in some cases where the growth hypotheses of lemmas 3.1 and 3.2 do not hold. Thus we can assert local existence near λ^* of two solutions in some cases where we do not know if there are two solutions on $[0, \lambda^*)$. For example, if $n = 3$, $m_0 = 5$ and $M_0 = 6$, then (3.34) holds but the growth conditions on f in lemmas 3.1 and 3.2 fail.

The inequality (3.28) can yield estimates on $\underline{u}_\lambda$ for various types of nonlinearity. The difficulty is in making the correct choices of G, ψ, and ϕ. The hypotheses of lemma 3.11 are not the only ones which can be used. For example, we may replace the assumptions (3.29), (3.30), with

$$f^\alpha(x, u) = g^\alpha(x)(u^\alpha)^m + h^\alpha(x, u) \tag{3.35}$$

and

$$|h^\alpha(x, u)| \leq \varepsilon|u|^m + C(1 + |u|^{m-\delta}) \tag{3.36}$$

where $m > 1$, $\delta > 0$, and the remaining assumptions previously imposed on f and h^α are still in force. Again, let $G^{\alpha\alpha} = mg^\alpha(x)(u^\alpha)^{m-1}$, $\phi^\alpha = (u^\alpha)^j$ and $\psi^\alpha = [j^2/(2j-1)](u^\alpha)^{2j-1}$.
Then for $u = \underline{u}_\lambda$, (3.28) yields

$$\begin{aligned}\lambda \int_\Omega \sum_{\alpha=1}^k \left[\frac{j^2}{(2j-1)}\right] g^\alpha (u^\alpha)^{2j+m-1}\,dx &\\ \geq \lambda \int_\Omega \sum_{\alpha=1}^k m g^\alpha (u^\alpha)^{2j+m-1}\,dx &\\ - \int_\Omega \sum_{\alpha=1}^k c^\alpha (u^\alpha)^{2j}\,dx &\\ + \int_\Omega \left[\frac{j^2}{(2j-1)}\right][c^\alpha - \lambda h^\alpha](u^\alpha)^{2j-1}\,dx &\end{aligned} \tag{3.37}$$

Using (3.36) we have

$$\begin{aligned}\int_\Omega h^\alpha (u^\alpha)^{2j-1}\,dx \leq \varepsilon \int_\Omega |u|^{2j+m-1}\,dx &\\ + C\int_\Omega (1 + |u|^{m-\delta})|u|^{2j-1}\,dx. &\end{aligned} \tag{3.38}$$

Let $g_0 = \min\{g^\alpha(x): x \in \bar{\Omega},\ \alpha = 1, \ldots, k\} > 0$, and let $V = \sum_{\alpha=1}^{k} g^\alpha(x)(u^\alpha)^{2j+m-1}$. Then $V \geqslant g_0 \sum_{\alpha=1}^{k} (u^\alpha)^{2j+m-1}$. Also, as in [15, p. 63], we have $|u|^2 \leqslant k^{(p-2)/p}\left[\sum_{\alpha=1}^{k} (u^\alpha)^p\right]^{2/p}$, so that

$$|u|^p \leqslant k^{(p-2)/2} \sum_{\alpha=1}^{k} (u^\alpha)^p \leqslant \left(\frac{k^{(p-2)/p}}{g_0}\right)V. \tag{3.39}$$

Thus, (3.37), (3.38) and (3.39) (with $p = 2j + m - 1$) yield

$$\lambda\left[\frac{j^2}{(2j-1)}\right]\left[1 + \varepsilon\left(\frac{k^{(2j+m-3)/2}}{g_0}\right)\right]\int_\Omega V\,\mathrm{d}x$$

$$\geqslant \lambda m \int_\Omega V\,\mathrm{d}x + \text{lower order terms} \tag{3.40}$$

where the lower order terms can be estimated in terms of the integral of $V^{1-\sigma}$ for some $\sigma > 0$. Then (3.40) yields a bound on V and hence on $\|u\|$ in $[L^{2j+m-1}(\Omega)]^k$ if

$$\left[\frac{j^2}{(2j-1)}\right]\left[1 + \varepsilon\left(\frac{k^{2j+m-3)/2}}{g_0}\right)\right] < m \tag{3.41}$$

holds. But (3.41) is satisfied if $j^2/(2j-1) < m$ and ε is sufficiently small. Arguing as in the discussion leading to lemma 3.11, we see that $M < \infty$ so we can apply corollary 2.4 provided that

$$n < \frac{2m[2 + 2\sqrt{\gamma} + \gamma]}{(m-1)}, \tag{3.42}$$

where $\gamma = 1 - (1/m)$, and ε is sufficiently small. (The size of ε depends on m, k, and g_0). Thus, the methods of [16] apply when there is coupling in the terms involving the highest powers of u^α occurring in the nonlinearity, provided the coupling coefficient is sufficiently small. Other structures for the nonlinearity may also be treated by these methods; for example the case of exponential nonlinearities for a single equation is discussed in [16]. We shall not attempt to make an exhaustive investigation of the circumstances under which the methods of [16] apply; the methods are rather flexible and appear to depend somewhat on the ingenuity of the investigator in choosing G, ψ and ϕ and in estimating the terms occurring in (3.28).

4. Qualitative properties of the solution set

In this section we prove some qualitative results about the solution set to (0.1). These include upper bounds on the number of solutions for given λ in certain situations and a description of the region in parameter space for which solutions exist in the multiparameter case.

First, let us impose on $f(x, u)$ the following weakened version of (H7):

$$\text{If } u > v \geqslant 0, \text{ then for all } \alpha, \beta \in \{1, \ldots, k\} \text{ and } x \in \Omega,\ \frac{\partial f^\alpha(x, u)}{\partial u^\beta} \geqslant \frac{\partial f^\alpha(x, v)}{\partial u^\beta} \tag{4.1}$$

with strict inequality for some α_0, $\beta_0 \in \{1, \ldots, k\}$ and $x_0 \in \Omega$.

We have the following:

PROPOSITION 4.1. If $f(x, u)$ satisfies (H1), (H2), (H5), (H6) and (4.1), then (0.1) cannot have three solutions $u_3 > u_2 > u_1 \gg 0$ for fixed λ.

Remark. This theorem is an extension to the case of systems of results obtained for a single equation in [21], [23].

Proof. Let

$$f_u(x, \phi, \psi) = \int_0^1 f_u(x, t\phi + (1-t)\psi)\, dt$$

so that $f(x, \phi) - f(x, \psi) = f_u(x, \phi, \psi)(\phi - \psi)$. Suppose that $u_3 > u_2 > u_1 \gg 0$ with u_1, u_2, u_3 solutions of (0.1) for some fixed λ. Let $w_1 = u_2 - u_1$, $w_2 = u_3 - u_2$. We have $w_1, w_2 > 0$, and satisfying

$$\begin{aligned} Lw_1 &= \lambda f_u(x, u_2, u_1) w_1 \quad &&\text{in } \Omega \\ w_1 &= 0 &&\text{on } \partial\Omega \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} Lw_2 &= \lambda f_u(x, u_3, u_2) w_2 \quad &&\text{in } \Omega \\ w_2 &= 0 &&\text{on } \partial\Omega. \end{aligned} \tag{4.3}$$

By our hypotheses on f, $f_u(x, u_2, u_1)$ and $f_u(x, u_3, u_2)$ are nonnegative irreducible with positivity on the diagonal on Ω, so by lemma 3.4 we have $w_1, w_2 \gg 0$. Since $tu_2 < tu_3$ and $(1-t)u_1 < (1-t)u_2$ for $t \in (0, 1)$, it follows from (4.1) that at least one entry of $f_u(x, u_3, u_2)$ is strictly larger than the corresponding entry of $f_u(x, u_2, u_1)$. Thus $[f_u(x, u_3, u_2) - f_u(x, u_2, u_1)]w_1$ is nonnegative and nontrivial. Combining (4.2) and (4.3) yields

$$\begin{aligned} L(w_2 - w_1) = {} & \lambda f_u(x, u_3, u_2)(w_2 - w_1) \\ & + \lambda[f(x, u_3, u_2) - f(x, u_2, u_1)]w_1 \end{aligned} \tag{4.4}$$

It follows from (4.3) and the fact that $w_2 \gg 0$ that λ is the principal eigenvalue for the system (4.3); so since $[f_u(x, u_3, u_2) - f_u(x, u_2, u_1)]w_1$ is nonnegative and nontrivial, it follows from lemma 2.1 of [11] that (4.4) has no nontrivial nonnegative solution. But then $0 \geq w_2(x_0) - w_1(x_0) = u_3(x_0) + u_1(x_0)$ for some $x_0 \in \Omega$, contradicting the hypothesis that $u_3 > u_1 \gg 0$; this contradiction proves the result.

Remark. The result is false if we eliminate the irreducibility hypothesis. Suppose that $f(x, u)$ and λ are such that the scalar equation $-\Delta u = \lambda f(x, u)$ in Ω, $u = 0$ on $\partial\Omega$, has solutions $\underline{u}_\lambda$ and $\bar{u}$ with $\bar{u} > \underline{u}_\lambda > 0$. (Sufficient conditions for this are derived in Sections 2 and 3.) Consider the uncoupled system

$$\begin{aligned} -\Delta u^1 &= \lambda f(x, u^1) \\ -\Delta u^2 &= \lambda f(x, u^2) \quad \text{in } \Omega \\ (u^1, u^2) &= (0, 0) \quad \text{on } \partial\Omega. \end{aligned} \tag{4.5}$$

The system (4.5) has solutions of the form $(\underline{u}_\lambda, \underline{u}_\lambda)$, $(\bar{u}, \underline{u}_\lambda)$, $(\underline{u}_\lambda, \bar{u})$, and $(\bar{u}, \bar{u})$. Since $(\bar{u}, \bar{u}) > (\underline{u}_\lambda, \bar{u}) > (\underline{u}_\lambda, \underline{u}_\lambda) \gg 0$, the theorem fails in this case.

We now consider the asymptotically linear case. Suppose that $f(x, u)$ is asymptotically linear and satisfies the conditions imposed at the beginning of this section. Suppose also that $f(x, u)$ is convex.

PROPOSITION 4.2. In the asymptotically linear case, the only solution to (0.1) for $\lambda < \lambda_\infty$ is $\underline{u}_\lambda$, the minimal solution.

Remark. This result is similar to proposition 3.2 of [4].

Proof. Suppose $\lambda < \lambda_\infty$ and $v \not\equiv \underline{u}_\lambda$ is a solution to (0.1). Then $v > \underline{u}_\lambda$. Also, we have

$$L(v - \underline{u}_\lambda) = \lambda f_u(x, v, \underline{u}_\lambda)(v - \underline{u}_\lambda).$$

Since $v - \underline{u}_\lambda > 0$ and $f_u(x, v, \underline{u}_\lambda)$ is nonnegative irreducible, it follows that $v - \underline{u}_\lambda \gg 0$. By hypothesis, $f(x, u)$ is convex and asymptotically linear, so $f_u(x, tv + (1-t)\underline{u}_\lambda) \leqslant f_u(x, \infty)$ entry by entry. Thus, $L(v - \underline{u}_\lambda) \leqslant \lambda f_u(x, \infty)(v - \underline{u}_\lambda)$ so $0 \ll v - \underline{u}_\lambda \leqslant \lambda L^{-1} f_u(x, \infty)(v - \underline{u}_\lambda)$. By lemma 3.4, there exists $\phi^* \in ([C_0^0(\bar{\Omega})]^k)^*$ such that $[L^{-1}f_u(x, \infty)]^*\phi^* = (1/\lambda_\infty)\phi^*$ and $\langle \phi^*, u\rangle > 0$ if $u \gg 0$. Thus we have $0 < \langle \phi^*, v - \underline{u}_\lambda\rangle \leqslant \lambda\langle \phi^*, L^{-1}f_u(x, \infty)(v - \underline{u}_\lambda)\rangle = (\lambda/\lambda_\infty)\langle \phi^*, v - \underline{u}_\lambda\rangle$. But $\lambda/\lambda_\infty < 1$, so this is impossible. Hence (0.1) cannot have any solution other than the minimal one in the asymptotically linear case when $\lambda < \lambda_\infty$.

As noted in Section 2, our results extended to the multiparameter setting (0.6). In the case of a single parameter, Λ is either $(0, \lambda^*)$ or $(0, \lambda^*]$ depending on the particular nonlinearity f involved. However, in the multiparameter case, Λ should be a region in the positive cone of $\mathbb{R}^k$. Interesting questions now arise concerning the shape of this region. We conclude our paper with a brief examination of some of the basic features of the set Λ in the multiparameter case. Our principal observation is contained in the following lemma.

LEMMA 4.3. Suppose (H1)–(H4) hold. Consider the system

$$\begin{aligned} L^\mu u^\mu &= \lambda^\mu f^\mu(x, u) && \text{in } \Omega. \\ u^\mu &> 0 && \text{in } \Omega \\ u^\mu &\equiv 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.6}$$

$\mu = 1, \ldots, k$. Let $\bar{\Lambda}$ denote the set $\{(\lambda^1, \ldots, \lambda^k)\colon \lambda^\mu > 0,\ \mu = 1, \ldots, k$, and (4.6) has solution $u \gg 0\}$. Then if $\bar{\lambda} = (\bar{\lambda}^1, \ldots, \bar{\lambda}^k) \equiv \bar{\Lambda}$, and $0 \ll \lambda < \bar{\lambda}$, $\lambda \in \bar{\Lambda}$.

Proof. Suppose $0 \ll \lambda < \bar{\lambda}$. Note that a solution to (4.6) at λ is equivalent to a solution of

$$\begin{aligned} L^\mu u^\mu &= c\bar{f}^\mu(x, u) && \text{in } \Omega, \mu = 1, \ldots, k \\ u &\gg 0 && \text{in } \Omega \\ u &\equiv 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.7}$$

where

$$c = \left(\sum_{\mu=1}^{k} (\lambda^{\mu})^2\right)^{\frac{1}{2}} \quad \text{and} \quad \tilde{f}^{\mu}(x, u) = \frac{\lambda^{\mu}}{c} f^{\mu}(x, u).$$

Let $\bar{u} = \underline{u}(\bar{\lambda}, x)$ denote the minimal positive solution to associated with $\bar{\lambda}$, and, as in theorem 3.1 of [11], let $u_0 \equiv 0$ and u_n be given by the iteration scheme

$$(L + cK)u_{n+1} = c(\tilde{f}(x, u_n) + Ku_n). \tag{4.8}$$

As is shown in [11], $\lambda \in \bar{\Lambda}$ provided the iterates of (4.8) are bounded above. But notice $\bar{u} - u_0 \geq 0$ and if $\bar{u} - u_n \geq 0$,

$$\begin{aligned}(L^{\mu} + cK)(\bar{u}^{\mu} - u^{\mu}_{n+1}) &= \bar{\lambda}^{\mu} f^{\mu}(x, \bar{u}) + cK\bar{u}^{\mu} - c(\tilde{f}^{\mu}(x, u_n) + Ku^{\mu}_n) \\ &\geq \lambda^{\mu} f^{\mu}(x, \bar{u}) - c\tilde{f}^{\mu}(x, u_n) + cK(\bar{u}^{\mu} - u^{\mu}_n) \\ &= c(\tilde{f}^{\mu}(x, \bar{u}) - \tilde{f}^{\mu}(x, u_n)) + cK(\bar{u}^{\mu} - u^{\mu}_n) \\ &\geq 0,\end{aligned}$$

and, hence, the result follows.

COROLLARY 4.4. Suppose (H1), (H2), (H5) and (H6) hold. Then if λ and $\bar{\lambda}$ are as in lemma 4.3, $\underline{u}(\lambda, x) \ll \underline{u}(\bar{\lambda}, x)$.

Proof. Let $\gamma = \min\{\lambda^1, \ldots, \lambda^k\}$. Then $\gamma > 0$ and for $\mu = 1, \ldots, k$, one has

$$\begin{aligned}L^{\mu}(\underline{u}^{\mu}(\bar{\lambda}, x) - \underline{u}^{\mu}(\lambda, x)) &= \bar{\lambda}^{\mu} f^{\mu}(x, \underline{u}(\bar{\lambda}, x)) - \lambda^{\mu} f^{\mu}(x, \underline{u}(\lambda, x)) \\ &\geq \gamma(f^{\mu}(x, \underline{u}(\bar{\lambda}, x)) - f^{\mu}(x, \underline{u}(\lambda, x)).\end{aligned}$$

Since $\lambda < \bar{\lambda}$, there is at least one $\mu \in \{1, \ldots, k\}$ for which the above inequality is strict, and, hence, for this μ, $u^{\mu}(\lambda, x) \ll u^{\mu}(\bar{\lambda}, x)$. It now follows that

$$\begin{aligned}L(\underline{u}(\bar{\lambda}, x) - \underline{u}(\lambda, x)) &\geq \gamma[f(x, \underline{u}(\bar{\lambda}, x)) - f(x, \underline{u}(\lambda, x))] \\ &= \gamma\left[\int_0^1 f_u(x, \theta\underline{u}(\bar{\lambda}, x) + (1 - \theta)\underline{u}(\lambda, x))\, \mathrm{d}\theta\right] \cdot (\underline{u}(\bar{\lambda}, x) - \underline{u}(\lambda, x)).\end{aligned}$$

By (H6) and the fact that $\underline{u}(\bar{\lambda}, x) \gg 0$ and $\underline{u}(\lambda, x) \gg 0$,

$$\left[\int_0^1 f_u(x, \theta\underline{u}(\bar{\lambda}, x) + (1 - \theta)\underline{u}(\lambda, x))\, \mathrm{d}\theta\right]$$

is irreducible for $x \in \Omega$. Since $\underline{u}(\bar{\lambda}, x) \geq \underline{u}(\lambda, x)$ and $\underline{u}(\bar{\lambda}, x) - \underline{u}(\lambda, x) \not\equiv 0$, irreducibility implies that $\underline{u}(\bar{\lambda}, x) \gg \underline{u}(\lambda, x)$.

COROLLARY 4.5. Suppose (H1), (H2), (H5) and (H6) hold. Let $0 \ll \lambda \in \bar{\Lambda}$ and let $\mu_1(\lambda) > 0$ be the unique number such that

$$\begin{aligned}L\psi &= \mu_1(\lambda)(\lambda \cdot f_u(x, \underline{u}(\lambda, x))\psi \quad &\text{in } \Omega \\ \psi &= 0 \quad &\text{on } \partial\Omega\end{aligned} \tag{4.9}$$

has a solution $\psi \geqslant 0$. If $\lambda', \lambda'' \in \tilde{\Lambda}$ and $0 \leqslant \lambda' < \lambda''$, then $\mu_1(\lambda') > \mu_1(\lambda'')$. In particular, if

$$L\psi = \lambda'' \cdot f_u(x, \underline{u}(\lambda'', x))\psi \tag{4.10}$$

where $\psi \geqslant 0$ and $\psi|_{\partial\Omega} \equiv 0$, then $(I - \lambda' \cdot L^{-1}f_u(x, \underline{u}(\lambda', x)))^{-1}$ exists for any $\lambda' \in \tilde{\Lambda}$ for which $0 \leqslant \lambda' < \lambda''$.

Proof. The result is an easy extension of corollary 4.2 of [11]. Suppose ψ is a solution of (4.9) corresponding to $\lambda = \lambda''$.
Then

$$\begin{aligned} L\psi = {} & \mu(\lambda'')(\lambda' \cdot f_u(x, \underline{u}(\lambda', x)))\psi \\ & + \mu(\lambda'')[\lambda'' \cdot f_u(x, \underline{u}(\lambda'', x)) - \lambda' \cdot f_u(x, \underline{u}(\lambda', x))]\psi. \end{aligned} \tag{4.11}$$

Since (H5) holds, $\lambda' < \lambda''$, and $\underline{u}(\lambda', x) \leqslant \underline{u}(\lambda'', x)$, the last term of the right-hand side of (4.11) is nonnegative and trivial. Hence the positivity lemma of [11] implies $\mu_1(\lambda'') < \mu_1(\lambda')$. If (4.10) obtains, $\mu_1(\lambda'') = 1$ and hence $\mu_1(\lambda') > 1$ for any $\lambda' \in \tilde{\Lambda}$ with $\lambda' < \lambda''$. Hence $I - \lambda \cdot L^{-1}f_\mu(x, \underline{u}(\lambda', x))$ is invertible.

We are now able to establish our main result on this topic. Namely, $\tilde{\Lambda}$ is a continuous deformation of the intersection of the unit sphere and positive cone in $\mathbb{R}^k$. Furthermore, in case f is convex and $\tilde{\Lambda}$ is closed, if one component of the tuple $(\lambda_1, \ldots, \lambda_k) \in \partial\tilde{\Lambda}$ increases, the other components necessarily decrease. For the sake of simplicity, we will prove the result only in the case $k = 2$.

Observe that the statement (u_1, u_2) is a solution to (4.6) for (λ_1, λ_2) is equivalent to the statement that (u_1, u_2) is a solution to

$$\begin{aligned} L^1u^1 &= \mu \cos\left(\frac{\pi}{2} - t\right) f^1(x, u^1, u^2) \\ L^2u^2 &= \mu \sin\left(\frac{\pi}{2} - t\right) f^2(x, u^1, u^2). \end{aligned} \tag{4.12}$$

for some $\mu > 0$ and $t \in (0, \pi/2)$. Let

$$f_t(x, u^1, u^2) = \begin{pmatrix} \cos(\pi/2 - t) f^1(x, u^1, u^2) \\ \sin(\pi/2 - t) f^2(x, u^1, u^2) \end{pmatrix}$$

and define

$$\mu^*(t) = \lambda^*(f_t) = \sup\{\lambda > 0 \colon Lu = \lambda f_t(x, u) \text{ has a positive solution}\}.$$

THEOREM 4.6. Consider (4.12). Assume f satisfies (H1), (H2), (H5) and (H6). Then $\mu^*\colon (0, \pi/2) \to (0, \infty)$ is continuous. If, in addition, f is convex and

$$(\mu^*(t)\cos(\pi/2 - t), \mu^*(t)\sin(\pi/2) - t)) \in \tilde{\Lambda},$$

for $t \in (0, \pi/2)$, then $t < t'$ implies

$$\mu^*(t)\cos(\pi/2 - t) < \mu^*(t')\cos(\pi/2 - t')$$

while

$$\mu^*(t)\sin(\pi/2 - t) > \mu^*(t')\sin(\pi/2 - t').$$

Proof. Let $t_0 \in 0, \pi/2)$. Consider $(\mu^*(t_0)\cos(\pi/2) - t_0), \mu^*(t_0)\sin(\pi/2 - t_0))$. If $t < t_0$, lemma 4.3 implies that $\mu^*(t)\sin(\pi/2 - t) \geq \mu^*(t_0)\ \sin(\pi/2 - t_0)$ and $\mu^*(t)\cos(\pi/2 - t) \leq \mu^*(t_0)\cos(\pi/2 - t_0)$. Similarly, if $t > t_0$, $\mu^*(t)\cos(\pi/2 - t) \geq \mu^*(t_0)\cos(\pi/2 - t_0)$ while $\mu^*(t)\sin(\pi/2 - t) \leq \mu^*(t_0)\sin(\pi/2 - t_0)$. Hence $\mu^*(t)\cos(\pi/2 - t) \to \mu^*(t_0)\cos(\pi/2 - t_0)$ and $\mu^*(t)\sin(\pi/2 - t) \to \mu^*(t_0)\sin(\pi/2 - t_0)$ as $t \to t_0$. Thus

$$(\mu^*(t))^2 = (\mu^*(t))^2\cos^2(\pi/2 - t) + (\mu^*(t))^2\sin^2(\pi/2 - t)$$
$$\to (\mu^*(t_0))^2\cos^2(\pi/2 - t_0) + (\mu^*(t_0))^2\sin^2(\pi/2 - t_0) = (\mu^*(t_0))^2.$$

Thus $\mu^*(t) \to \mu^*(t_0)$ as $t \to t_0$. That the above inequalities are strict in case f is convex and $(\mu^*(t)\cos(\pi/2 - t), \mu^*(t)\sin(\pi/2 - t)) \in \bar{\Lambda}$ for $t \in (0, \pi/2)$ is a consequence of corollary 4.5 and the continuation arguments of Section 2.

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